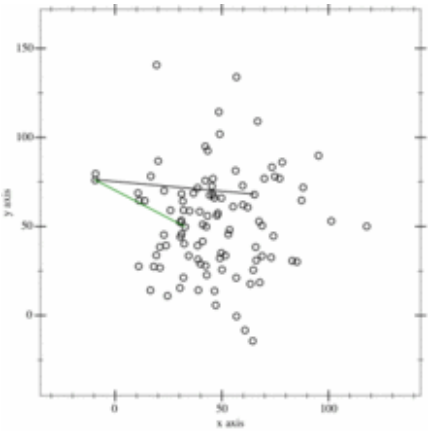
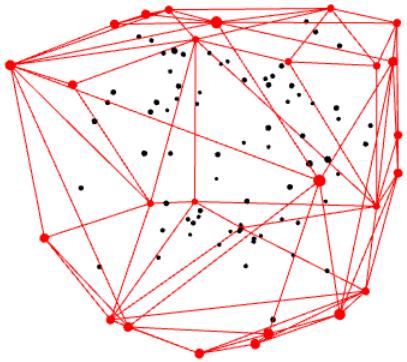
Computational Geometry

- Range queries
- Convex hulls
- Lower bounds
- Planar subdivision search
- Line segment intersection
- Convex polygons
- Voronoi diagrams
- Minimum spanning trees
- Nearest neighbors
- Triangulations
- Collinear subsets



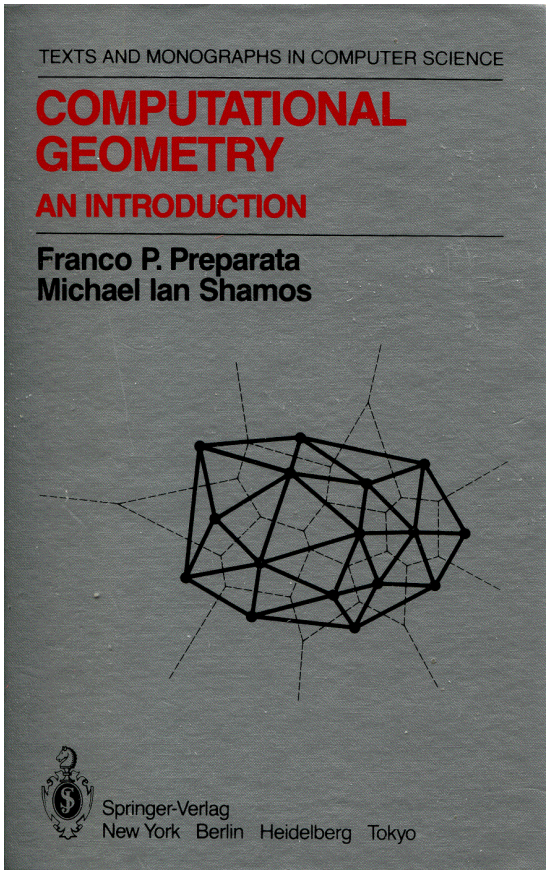


Computational Geometry

- "Algorithmic Geometry"
- Supplemental reading:

  Computational Geometry,

  An Introduction, by Franco Preparata and Michael Shamos, 1985
- Shamos founded Computational Geometry in mid-1970's with his Ph.D. thesis
- Now multiple conferences and journals
- Fundamental for graphics, gaming, CAD, motion planning, GIS / GPS,



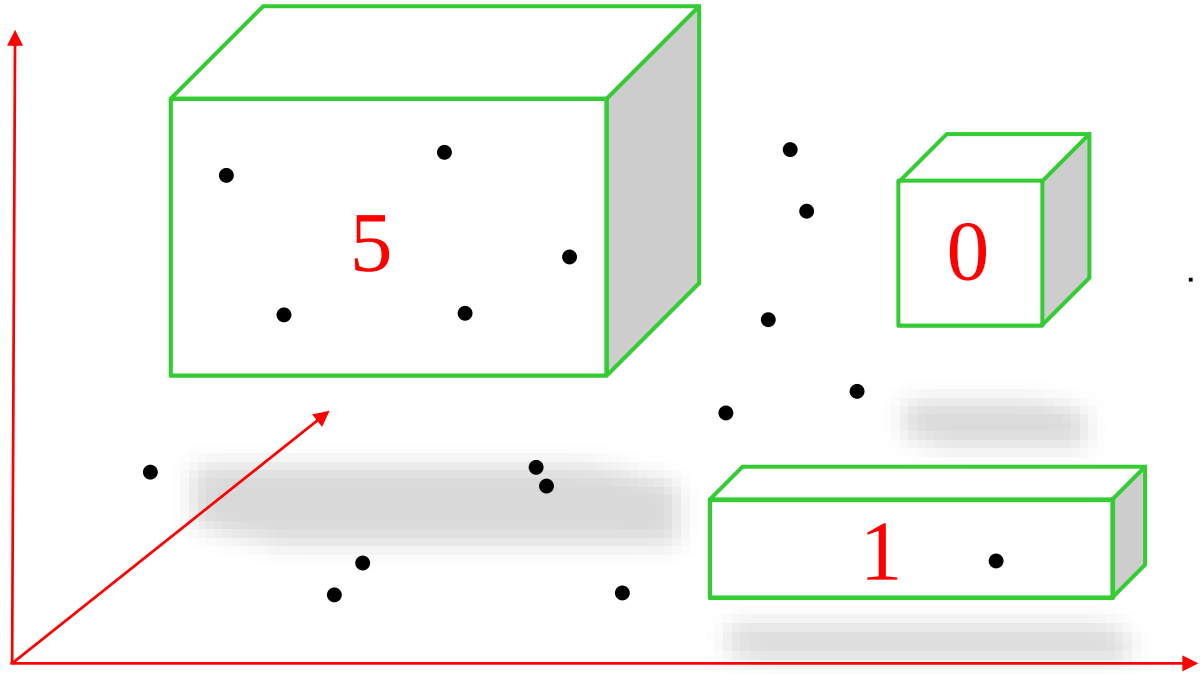
Michael Shamos

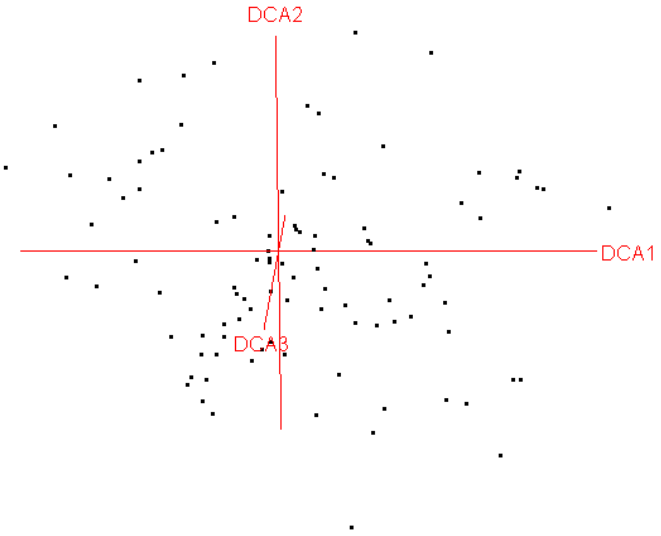
Franco Preparata

Range Queries

Input: n points (vectors), with preprocessing allowed

Output: number of points within any query rectangle





- Single preprocessing phase, but many queries
- Query time is more critical than preprocessing time
- Range queries generalize to any dimension

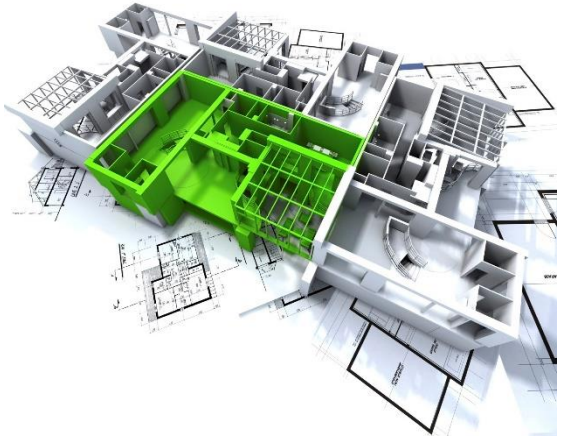
Range Queries

Input: n points (vectors), with preprocessing allowed

Output: number of points within any query rectangle

Applications:

- Databases
- GIS / GPS systems
- Gaming
- CAD



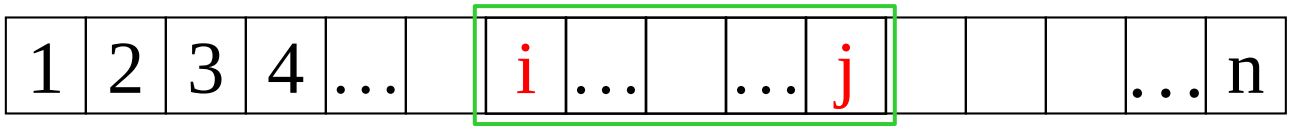
Range Queries

Input: n points (vectors), with preprocessing allowed

Output: number of points within any query rectangle

Idea: first solve the one-dimensional case!

- 1-D "rectangle" is a linear range / segment
- Preprocessing: sort input data



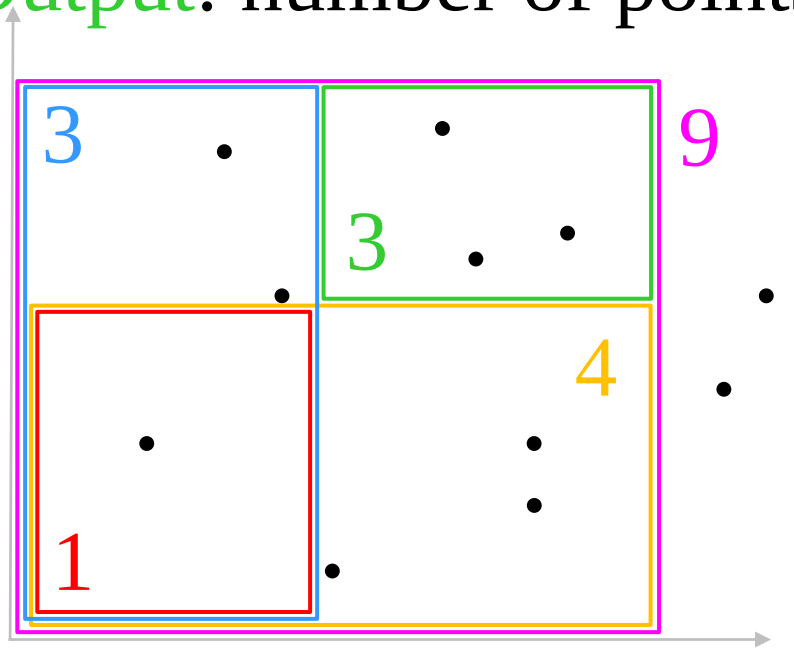
- Range query is a pair of binary searches
- O(log n) time per query
- O(n) space and O(n log n) preprocessing time

Q: Generalization to 2D?

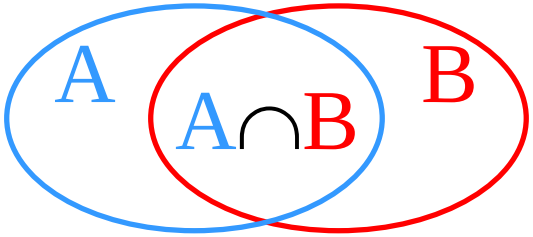
Range Queries

Input: n points (vectors), with preprocessing allowed

Output: number of points within any query rectangle



Inclusion-exclusion principle:



$$|A \cup B| = |A| + |B| - |A \cap B|$$

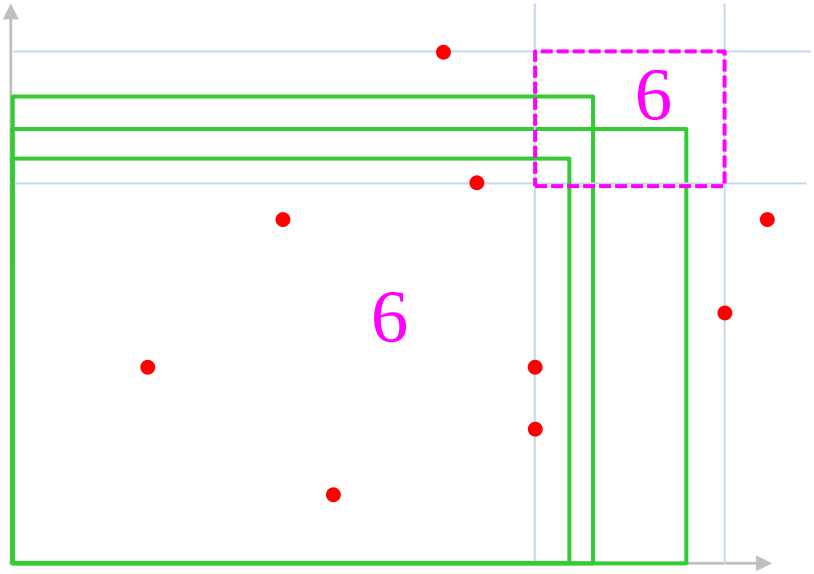
Idea: suffices to solve for origin-based rectangles

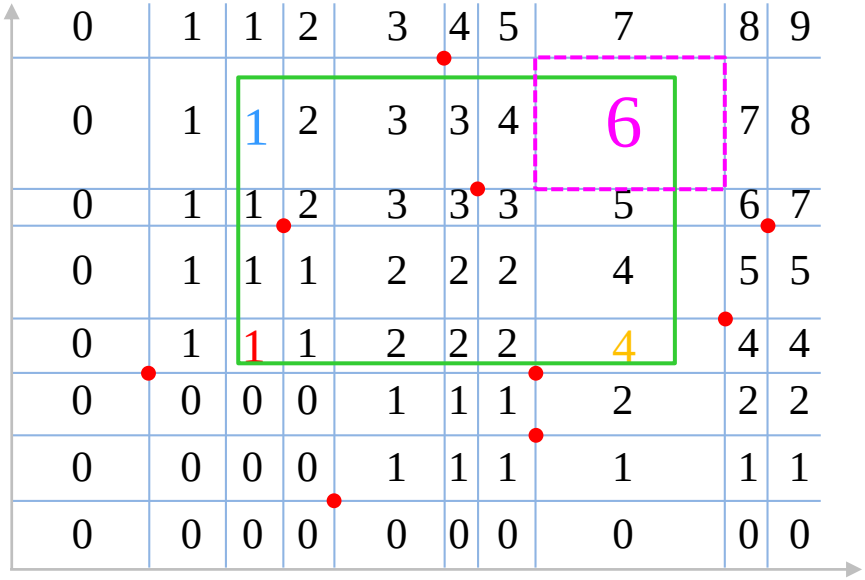
$\Rightarrow$ four calls to these solves the general case!

Range Queries

Input: n points (vectors), with preprocessing allowed

Output: number of points within any query rectangle





Idea: precompute south-west counts for all regions

- Use four binary searches to find four values
- Example: 6 – 1 – 4 + 1 = 2
- O(log n) time per query
- $O(n^2)$ space and $O(n^2)$ preprocessing time

Vector dominance!

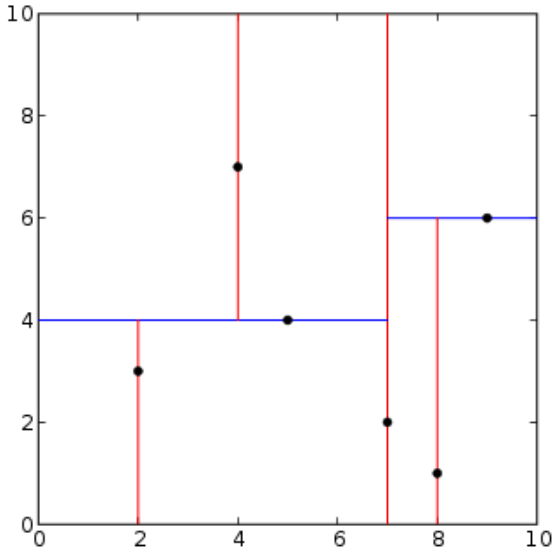
Range Queries

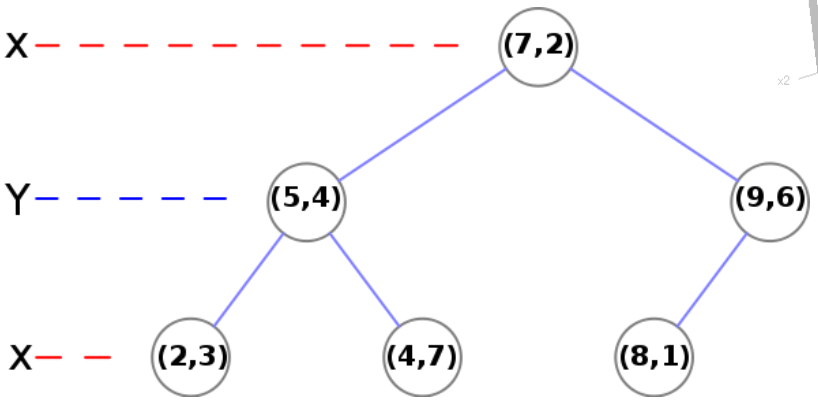
Input: n points (vectors), with preprocessing allowed

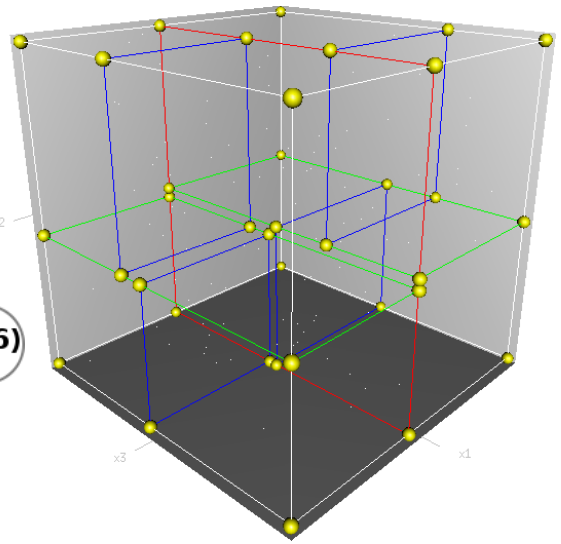
Output: number of points within any query rectangle

Other possible space-time tradeoffs:

	Preprocessing	space	query
Naïve:	O(1)	O(1)	O(n)
k-d trees:	O(n log n)	O(n log n)	$O(\log^2 n)$
Clever:	$O(n^2)$	$O(n^2)$	O(log n)



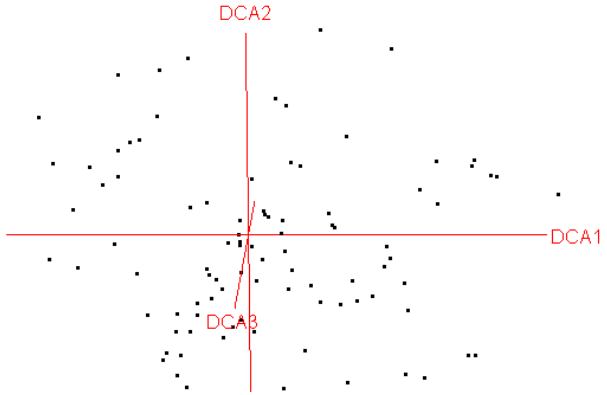


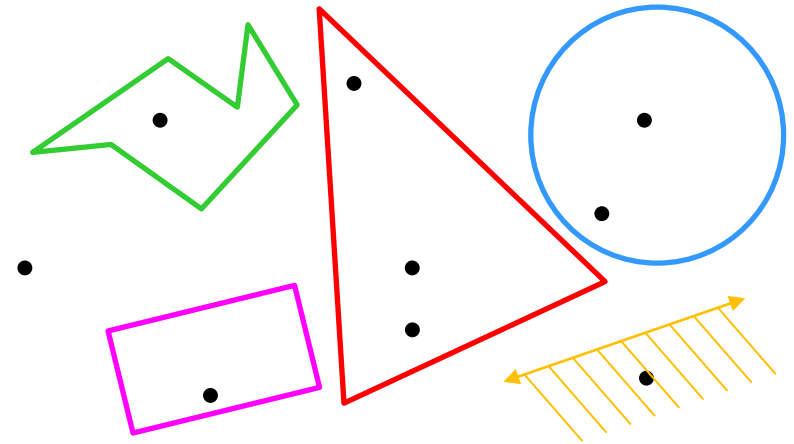


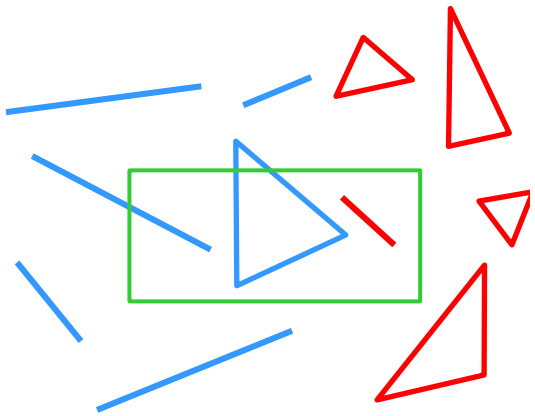
Range Queries

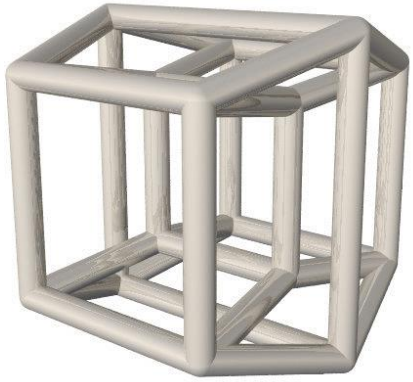
Generalizations:

- Higher dimensions
- General search window (not rectangular)
- Arbitrary objects (segments, polygons, mixed, etc.)
- Counting vs. reporting
- Containment vs. intersection
- Static vs. dynamic
- Online vs. offline





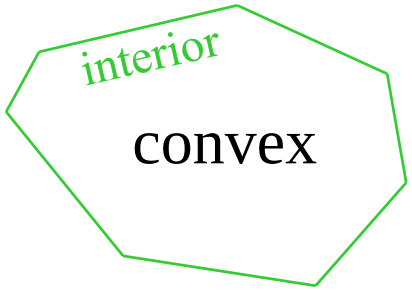


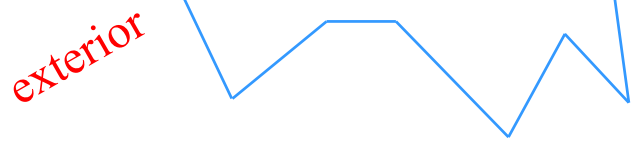


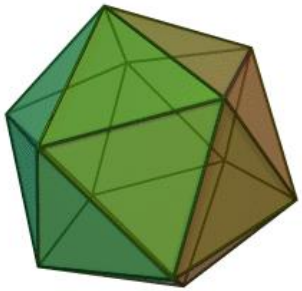
Polygons

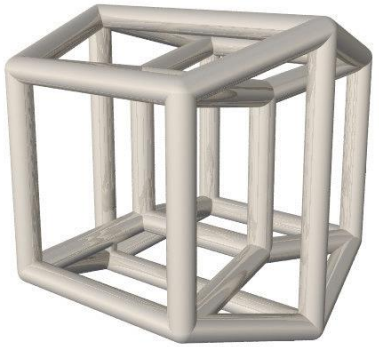
Definition: a polygon is a closed sequence of vertices

Definition: a simple polygon has no self-intersections









Theorem [Jordan]: a simple polygon partitions the plane into 3 regions: interior, exterior, and boundary

Definition: convex polygon contains all pairwise segments (i.e. is the intersection of half-planes)

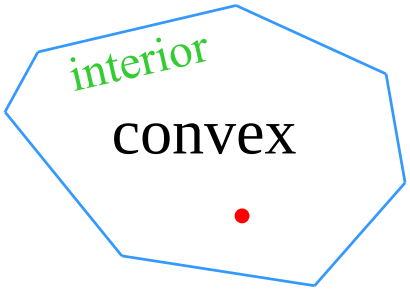
Definition: convex polygon is the intersection of all the half-planes containing its vertices

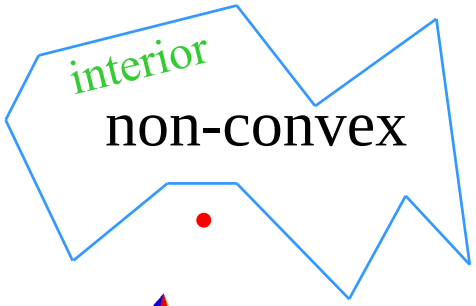
Definition: a polytope is a higher-dimensional polygon

Point Location

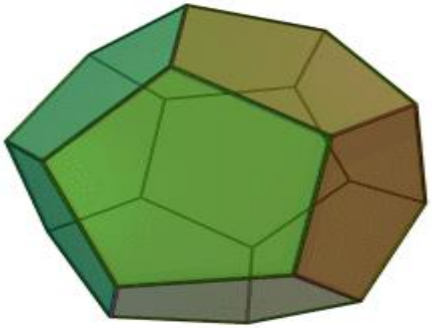
Input: polygon and query point

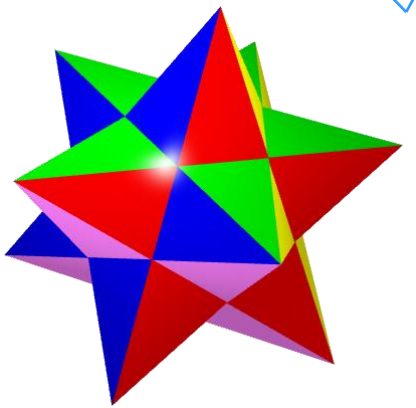
Output: is query point interior to polygon?

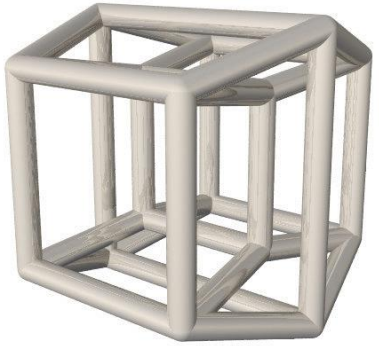




exterior







- Single preprocessing phase, but many queries
- Query time is more critical than preprocessing time
- Point location problem generalizes to any dimension

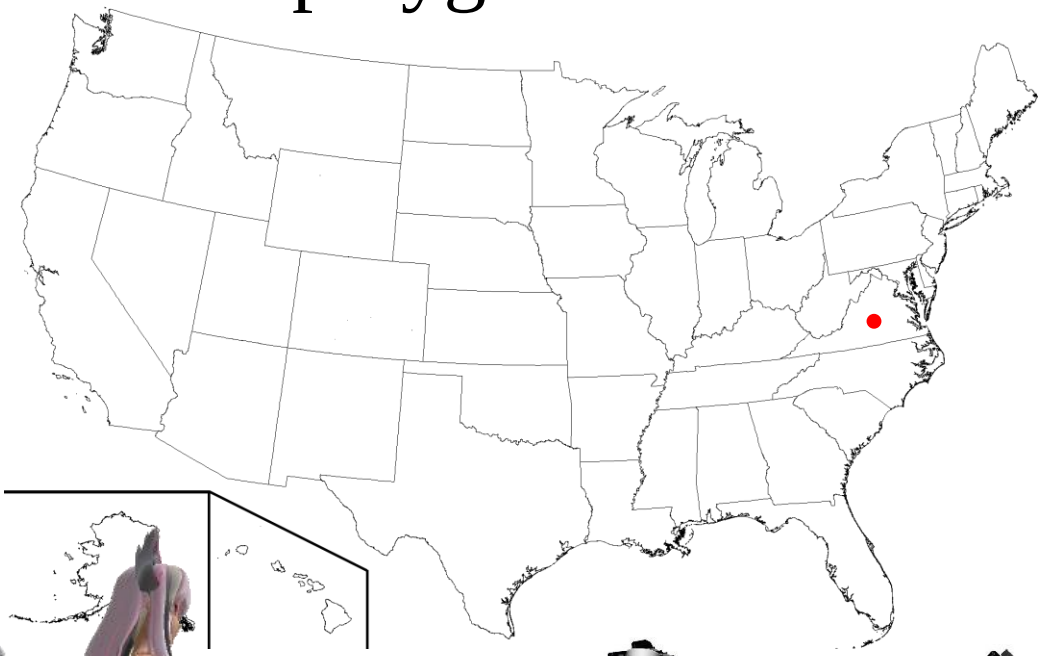
Point Location

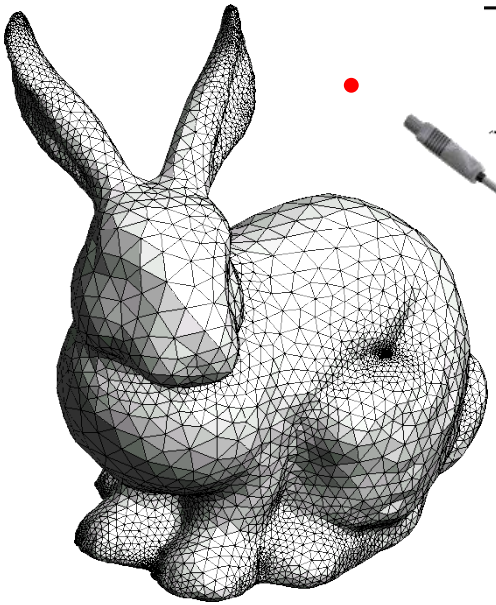
Input: polygon and query point

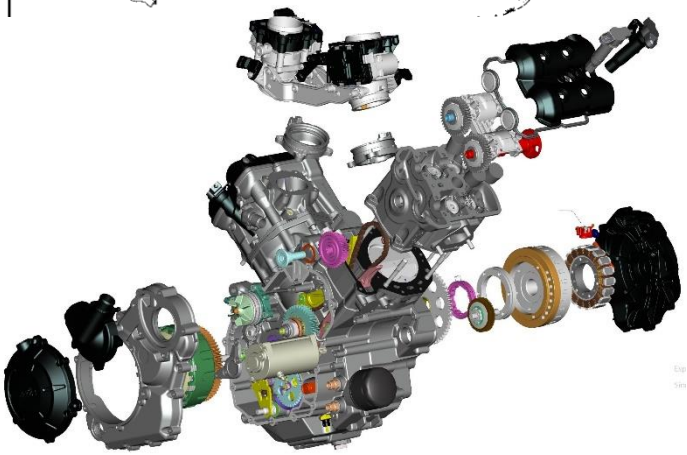
Output: is query point interior to polygon?

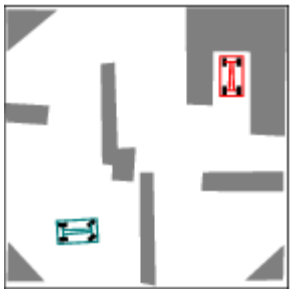
Applications:

- Mouse clicking
- GIS / GPS systems
- Motion planning
- CAD
- Graphics
- Gaming





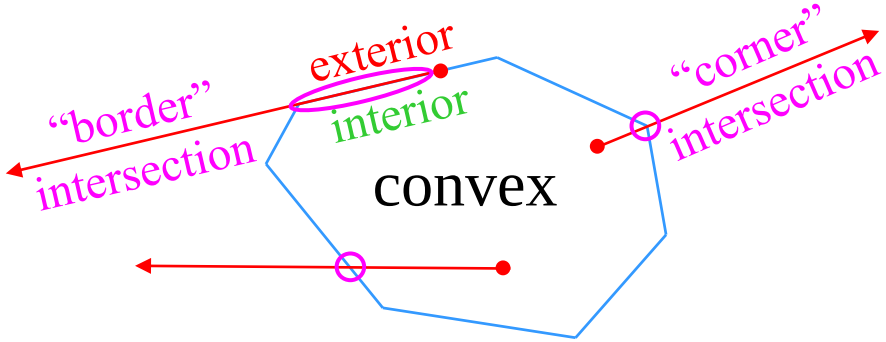


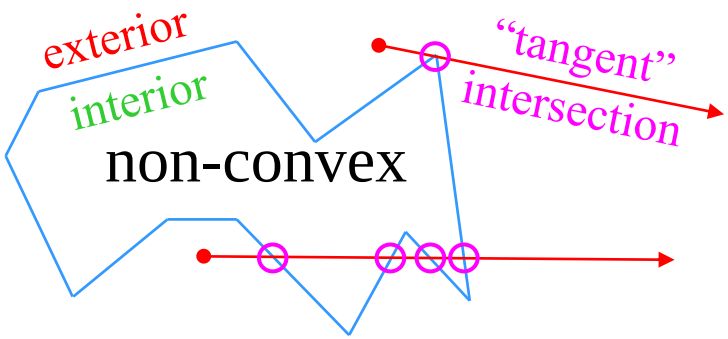


Point Location

Input: polygon and query point

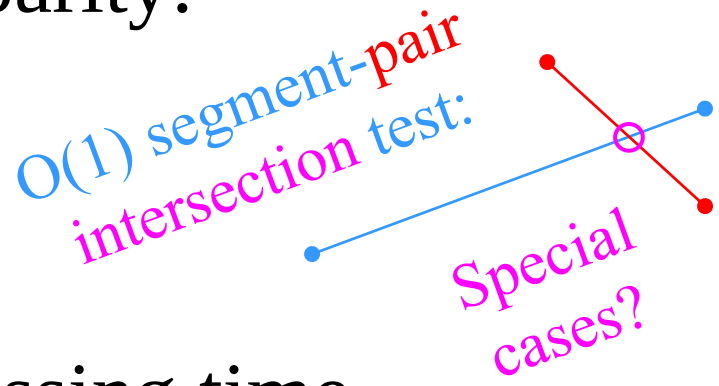
Output: is query point interior to polygon?





"Raycasting" algorithm (based on Jordan's theorem):

- Consider ray from query point to infinity
- Count segment intersections parity:
  - odd ⇒ interior
  - even ⇒ exterior
- O(n) time per query
- O(1) space and O(1) preprocessing time



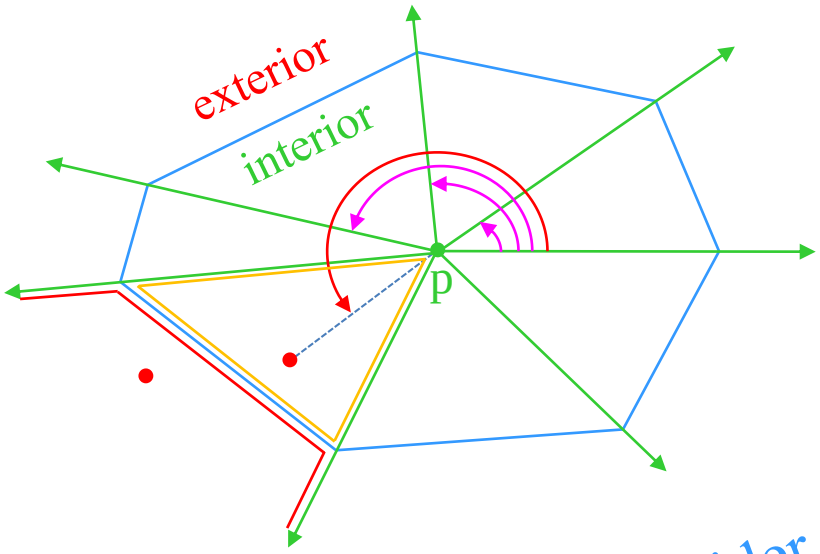
Point Location

Input: polygon and query point

Output: is query point interior to polygon?

Convex case:

Preprocessing:
- Find an interior point p
- Partition into wedges w.r.t p
- Sort wedges by polar angle



Query:
- Find containing wedge
- Test interior/exterior

Need to consider finite numerical precision!

- O(log n) time per query (binary search)
- O(n) space and O(n log n) preprocessing time

Point Location

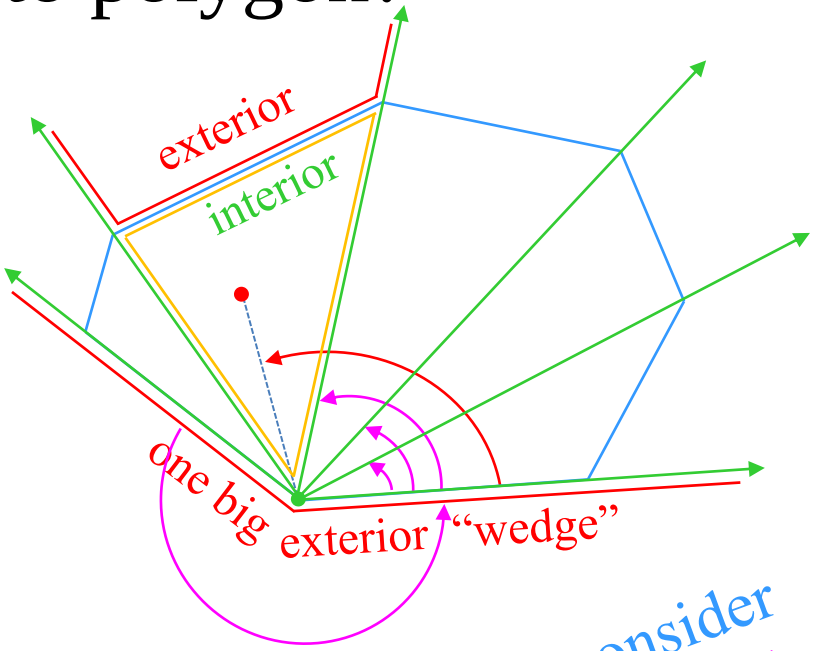
Other wedge schemes?

Input: polygon and query point

Output: is query point interior to polygon?

Convex case:

Preprocessing:
- Find an interior point p
- Partition into wedges w.r.t p
- Sort wedges by polar angle



Query:
- Find containing wedge
- Test interior/exterior

Need to consider finite numerical precision!

- O(log n) time per query (binary search)
- O(n) space and O(n log n) preprocessing time

Point Location

Non-convex case:

Input: polygon and query point

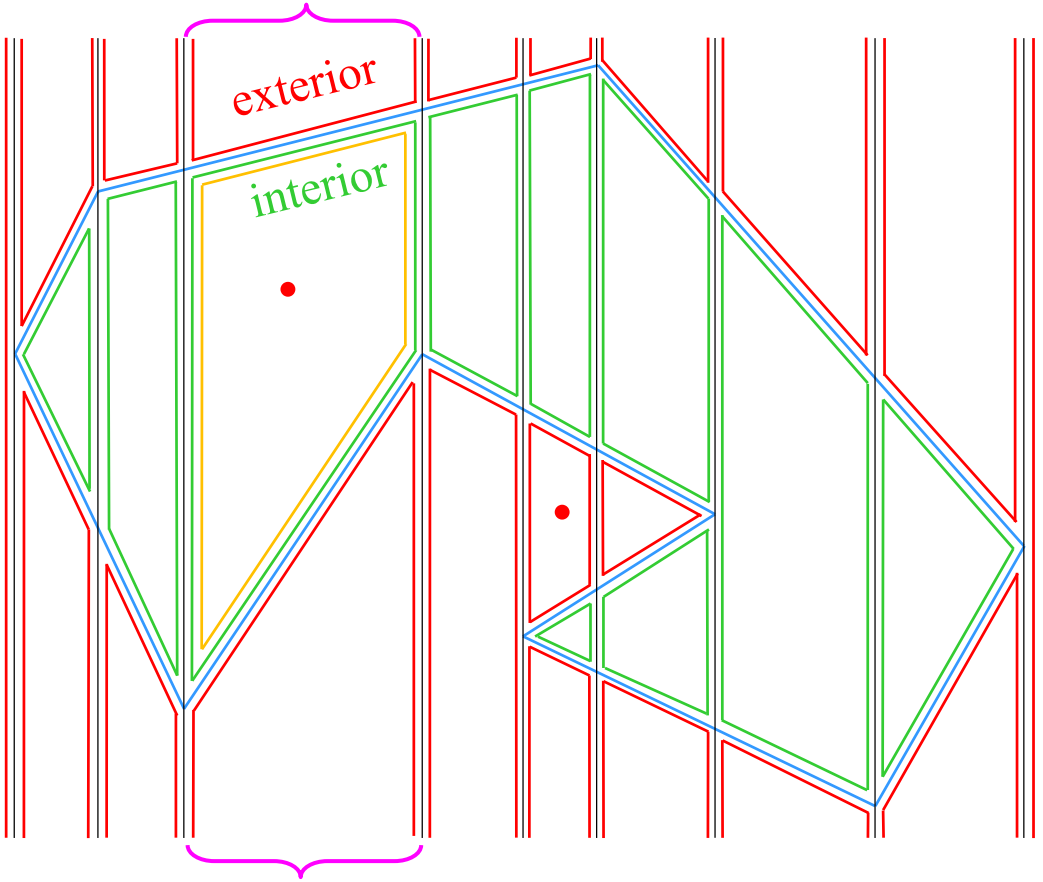
Output: is query point interior to polygon?

Preprocessing:

- Sort vertices by x
- Find vertical slices
- Partition into trapezoids
- Sort slice trapezoids by y

Query:

- Find containing slice
- Find trapezoid in slice
- Report interior/exterior



- $O(\log n)$ time per query (two binary searches)
- $O(n^2)$ space and $O(n^2)$ preprocessing time

Planar Subdivision Search

Arbitrary set of polygons

Input: polygon and query point

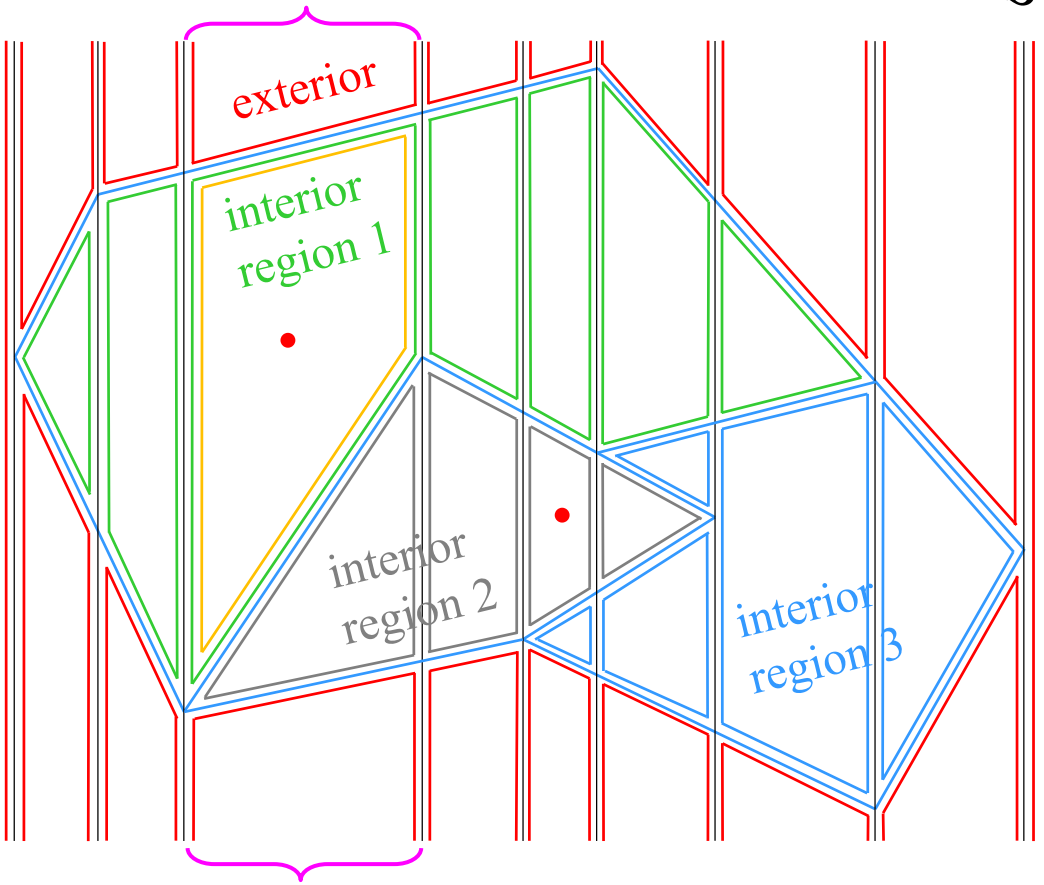
Output: is query point interior to polygon?

Preprocessing:

- Sort vertices by x
- Find vertical slices
- Partition into trapezoids
- Sort slice trapezoids by y

Query:

- Find containing slice
- Find trapezoid in slice
- Report interior/exterior



- O(log n) time per query (two binary searches)
- $O(n^2)$ space and $O(n^2)$ preprocessing time

Planar Subdivision Search

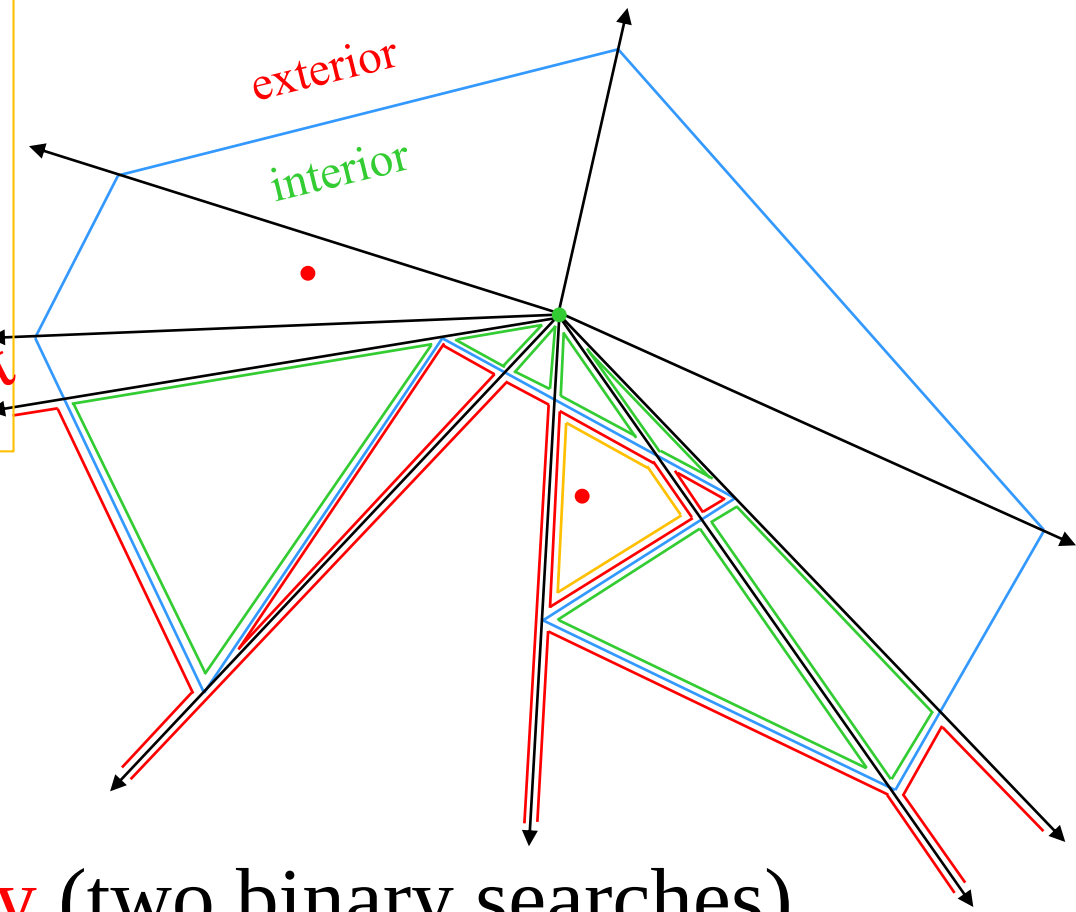
Other slicing Schemes:

Input: polygon and query point

Output: is query point interior to polygon?

Preprocessing:

- Sort vertices by x angle
- Find vertical slices
- Partition into trapezoids
- Sort slice trapezoids by y dist



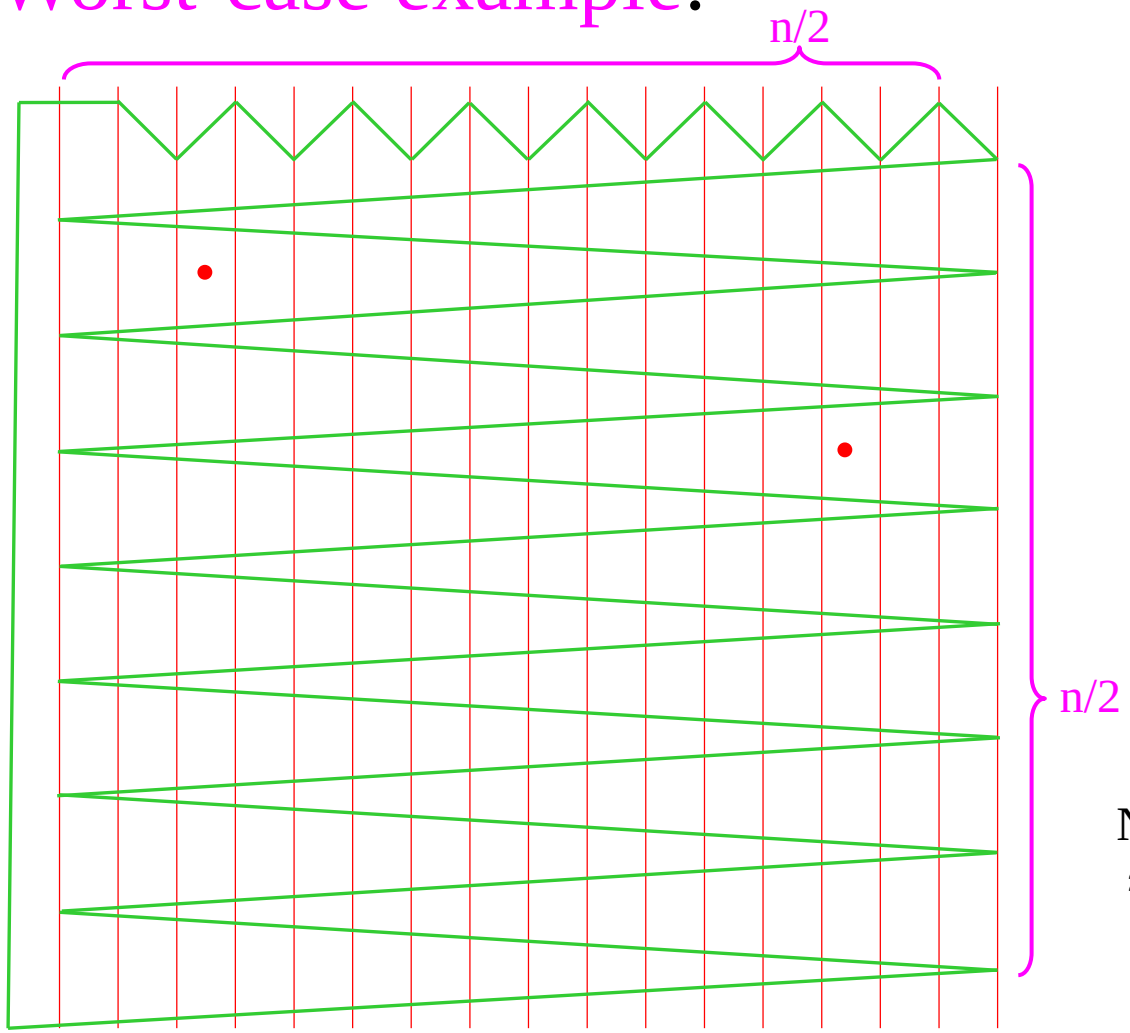
Query:

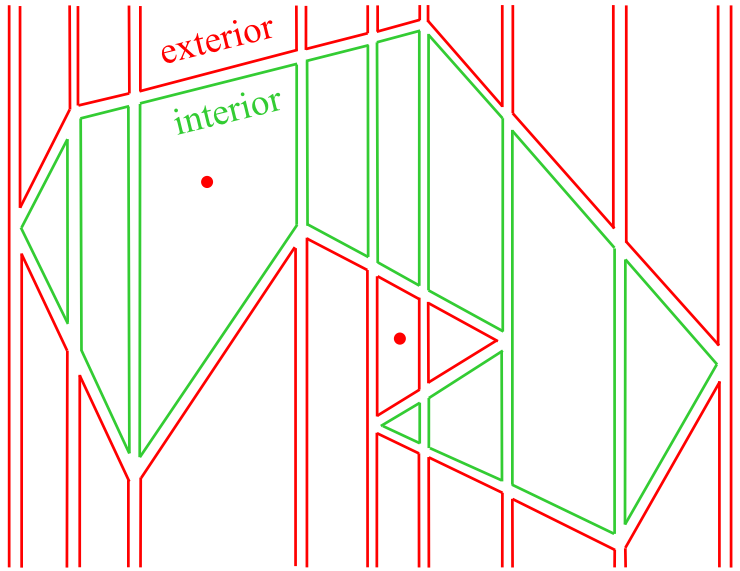
- Find containing slice
- Find trapezoid in slice
- Report interior/exterior

- O(log n) time per query (two binary searches)
- $O(n^2)$ space and $O(n^2)$ preprocessing time

Planar Subdivision Search

Worst-case example:

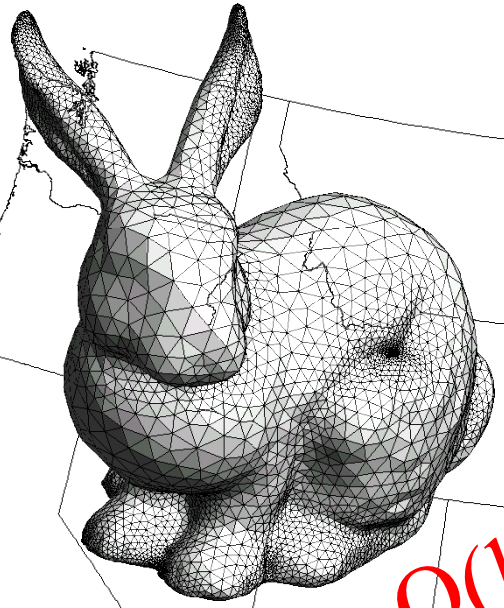




Number of subregions:
$\approx (n/2)(n/2) = n^2/4 = \Theta(n^2)$

- $\Theta(n^2)$ space and $\Theta(n^2)$ preprocessing time worst-case
- $O(n)$ space $O(n \log n)$ preproc $O(\log n)$ query possible!

Planar Subdivision Search



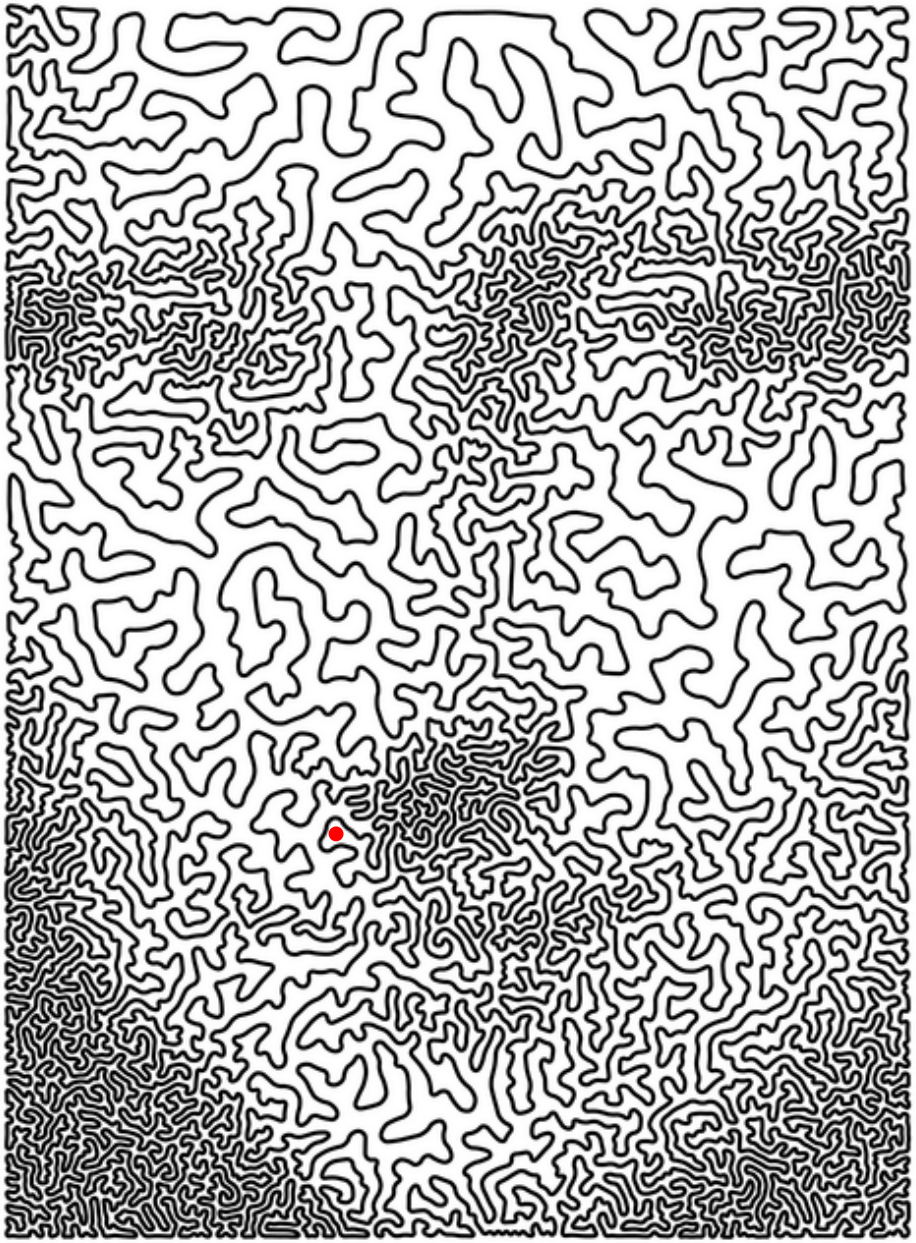
$O(\log n)$ time per query

$O(n^2)$ space, $O(n^2)$ preprocessing, or

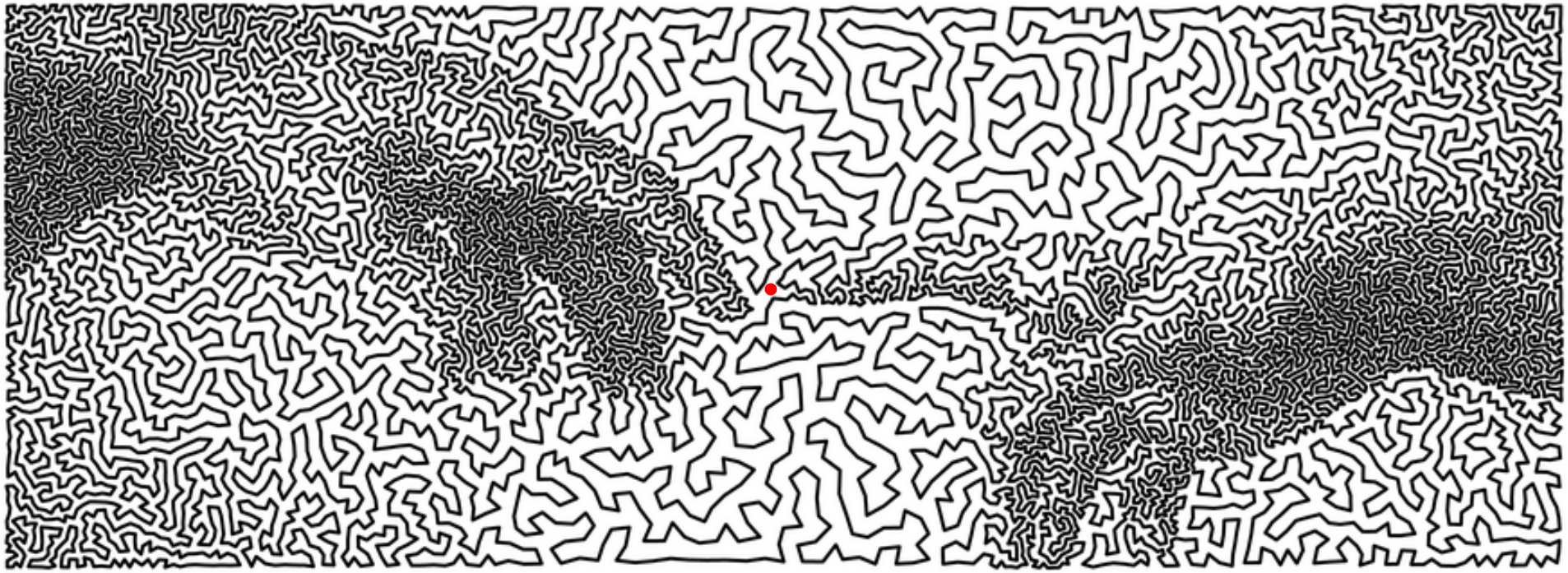
$O(n)$ space, $O(n \log n)$ preprocessing

non-contiguous

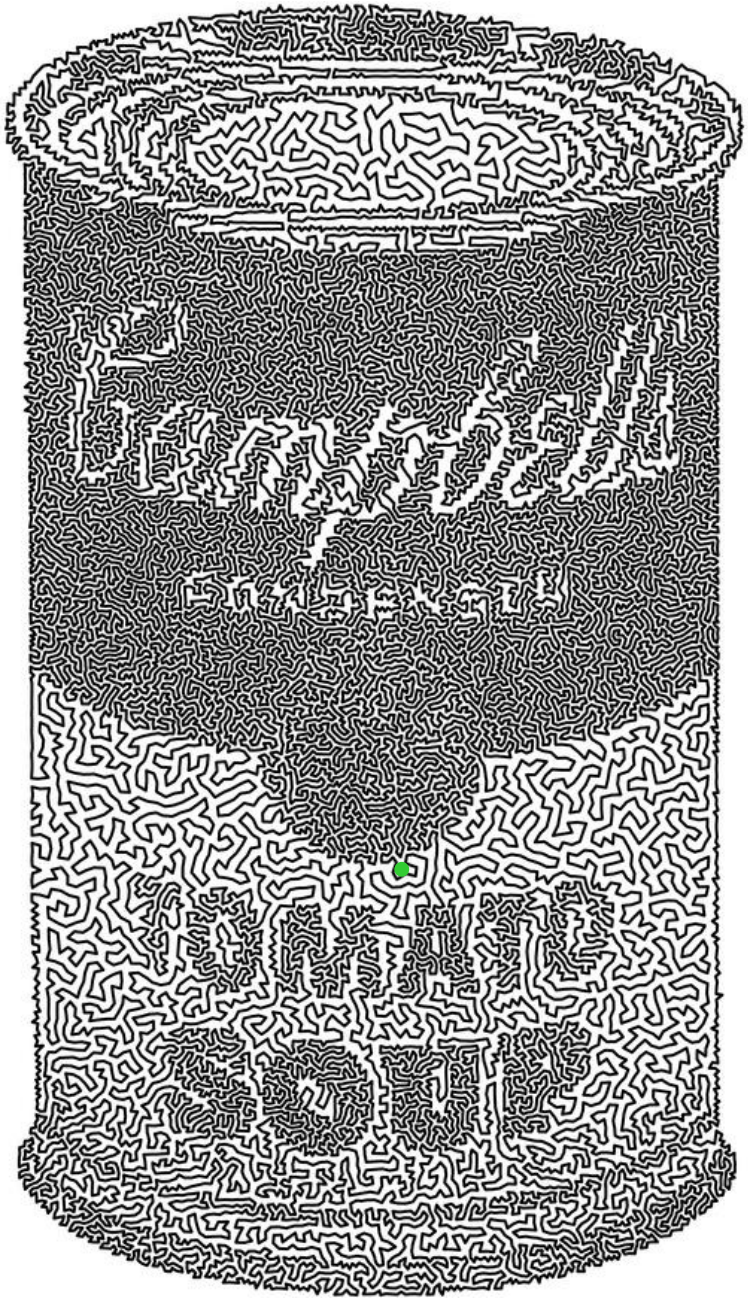
A “Simple” Polygon



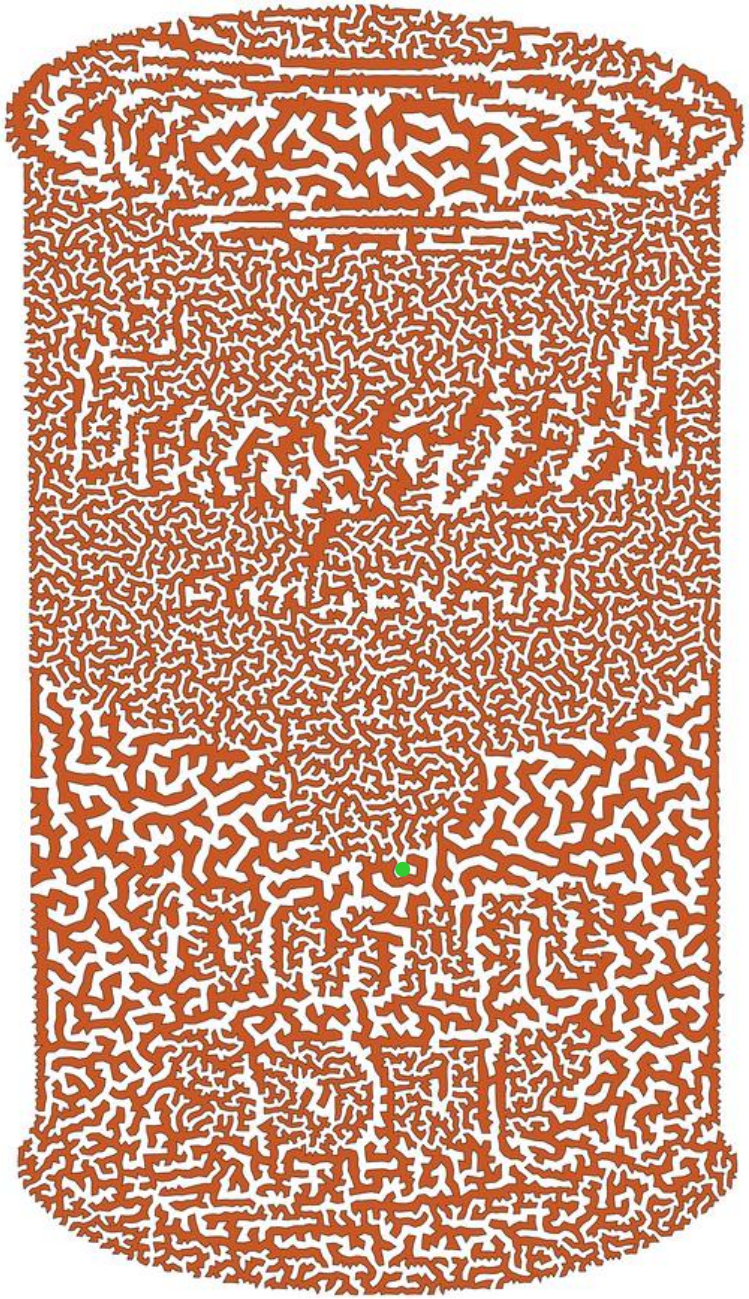
A “Simple” Polygon



A “Simple” Polygon



A “Simple” Polygon



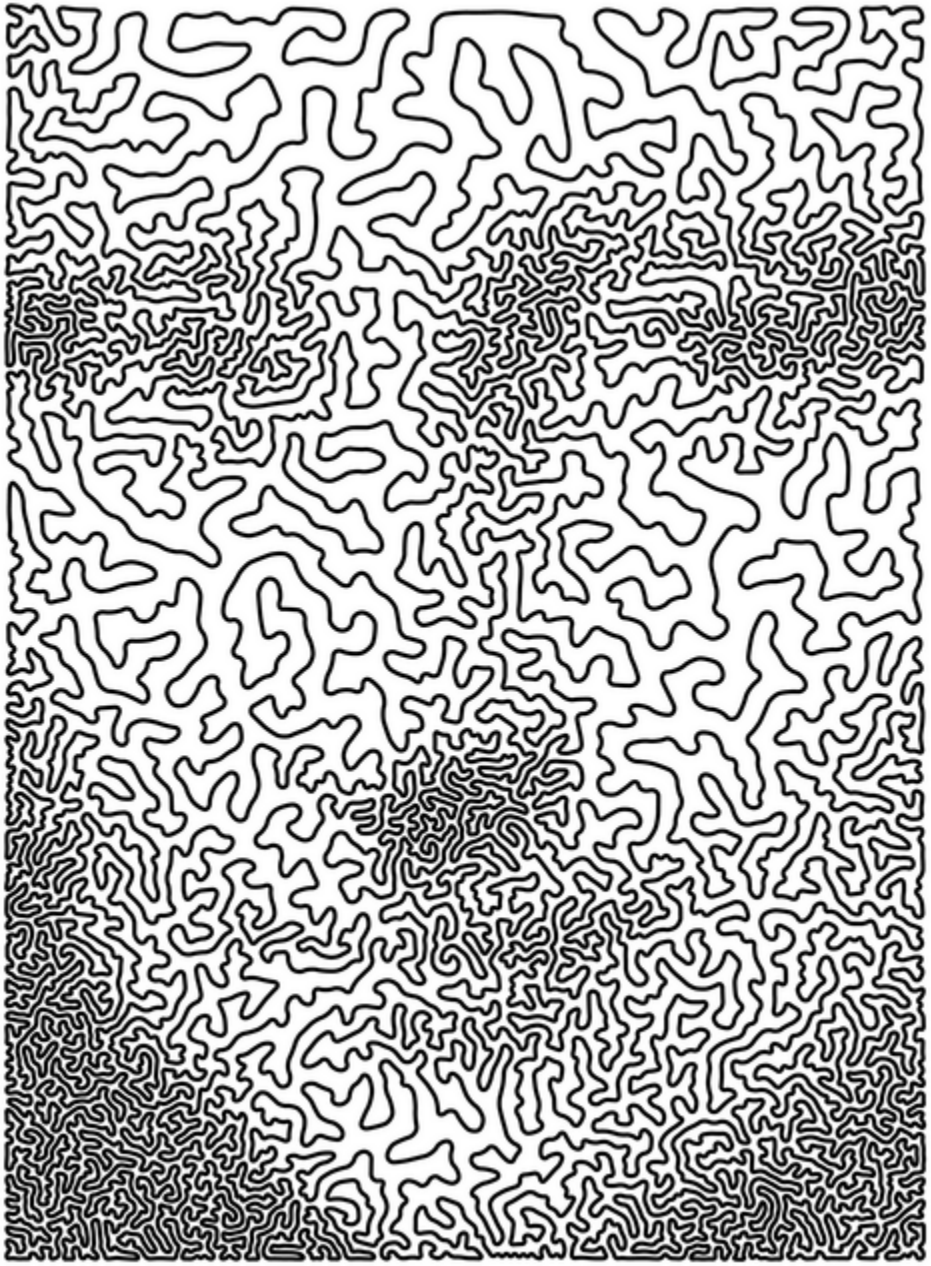
Project Idea: TSP Art

- Traveling Salesperson Tour
- Optimal is NP-complete

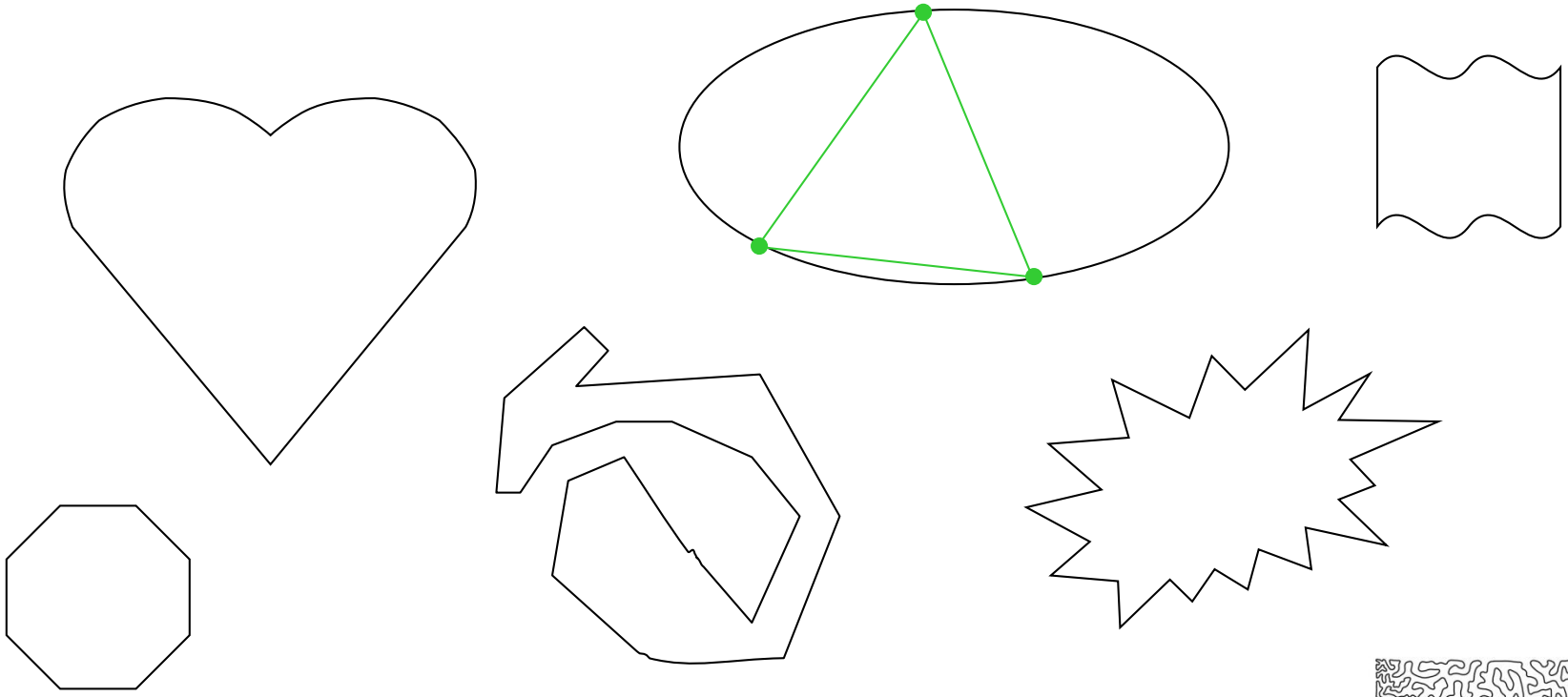
  $\Rightarrow$ use TSP heuristics

- Convert image to B&W
- Sample image density
- Create pointset
- Run TSP heuristics
- Uncross intersections

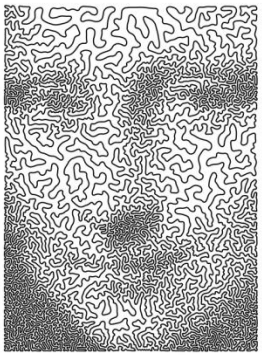
- Can use Minimum Spanning Trees (easy to compute)
- Can also use Minimum Matchings (easy to compute)
- What about colors?



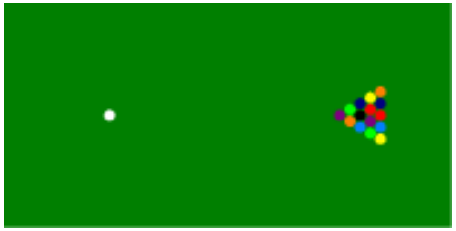
Problem: Does every closed simple curve contain the vertices of an equilateral triangle?



- What approaches fail?
- What techniques work and why?
- Lessons and generalizations

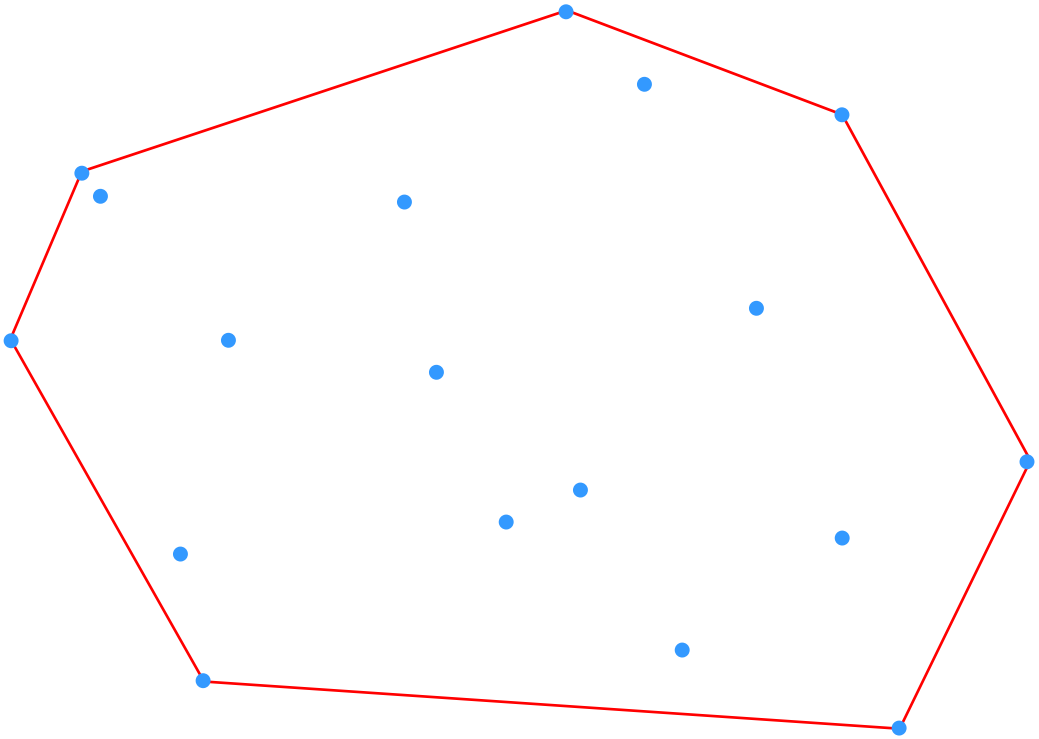


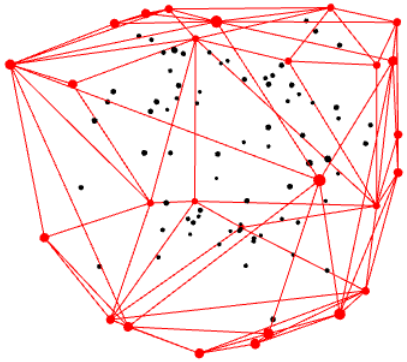
Convex Hulls

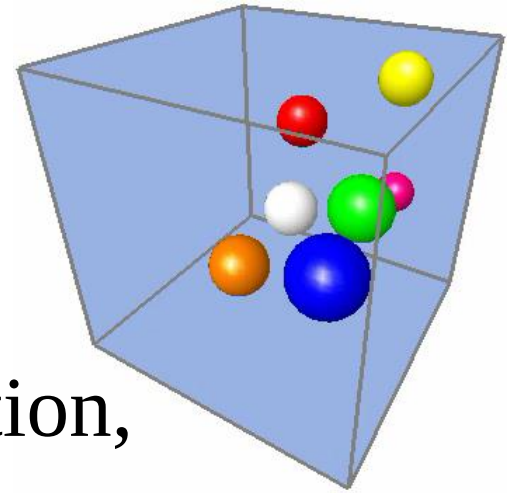


Input: set of n points

Output: smallest containing convex polygon







- Generalizes to higher dimensions
- Applications: gaming, collision detection, graphics, statistics, image recognition, …

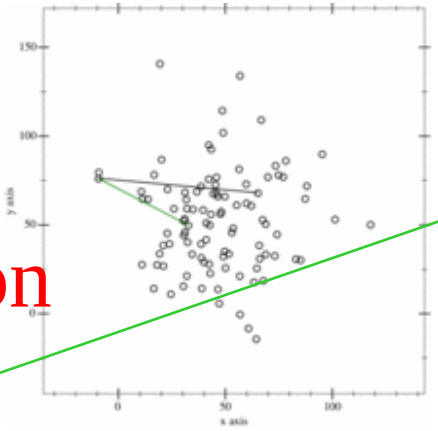
Convex Hulls

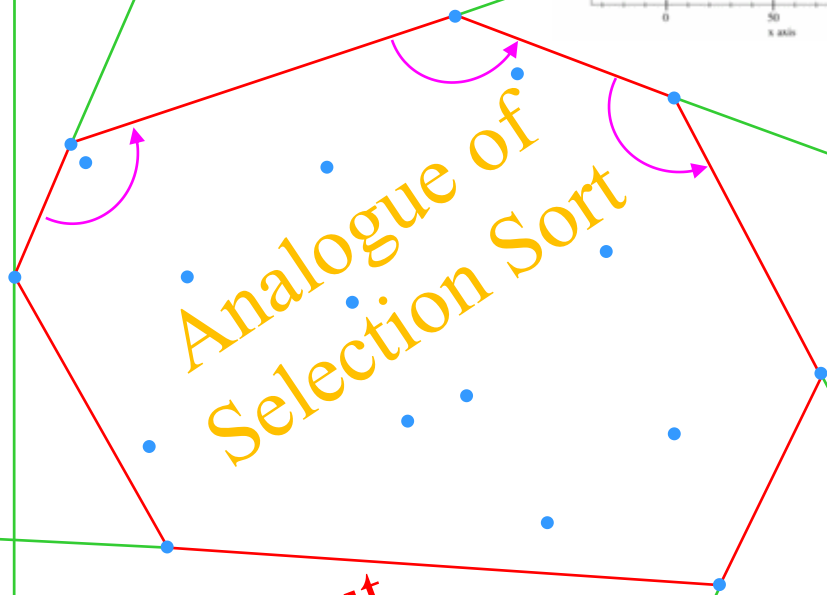
Input: set of n points

Output: smallest containing convex polygon

"Jarvis' march" [1973]:

- Start at point with least x
- Until CH is complete:
  - Find next CH point (with max internal angle)

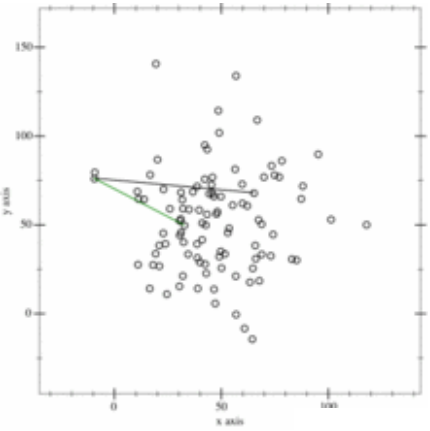




- $h \leq n$ convex hull points
- O(n) time per CH point
- $O(n \cdot h)$ time, $O(n^2)$ worst-case
- Generalizes to higher dimensions
- Parallelizes

Output sensitive!

Convex Hulls

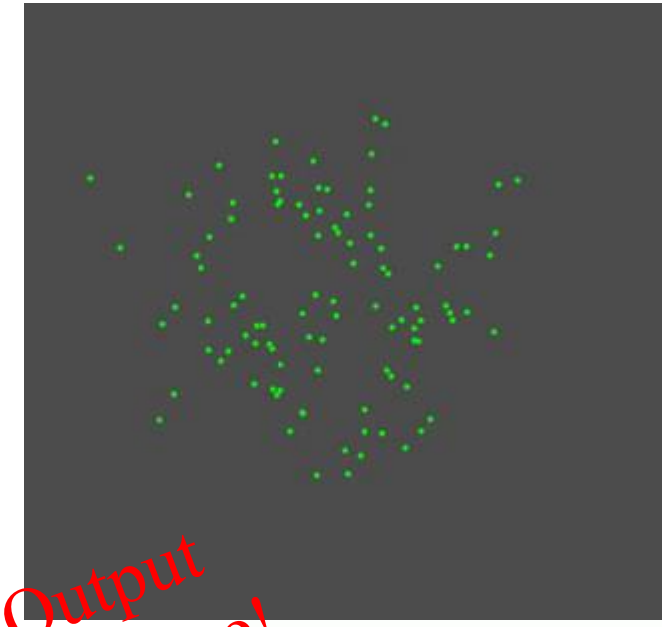


Input: set of n points

Output: smallest containing convex polygo

"Jarvis' march" [1973]:

- Start at point with least x
- Until CH is complete:
  - Find next CH point (with max internal angle)



- $h \leq n$ convex hull pc
- O(n) time per CH pc
- $O(n \cdot h)$ time, $O(n^2)$ worst-case
- Generalizes to higher dimensions
- Parallelizes

Output sensitive!

Convex Hulls

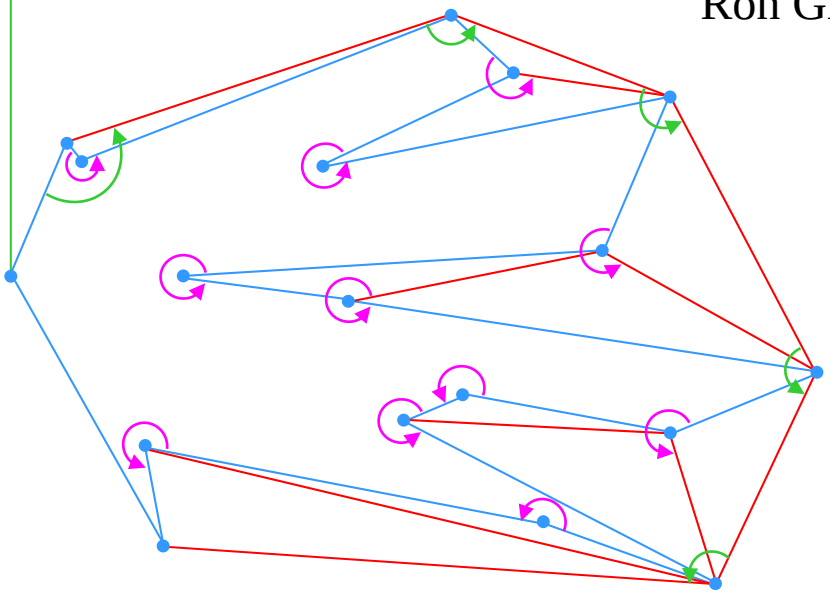

Ron Graham

Input: set of n points

Output: smallest containing convex polygon

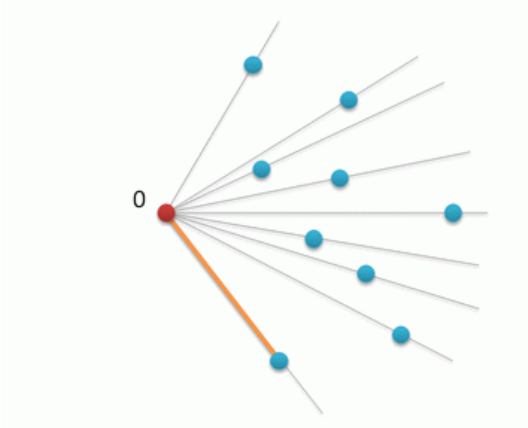
"Graham's scan" [1972]:

- Start at point with least x
- Sort points by polar angles
- Form star-shaped polygon
- Until CH is complete:
  - Scan next CH candidate
  - If reflex angle backtrack



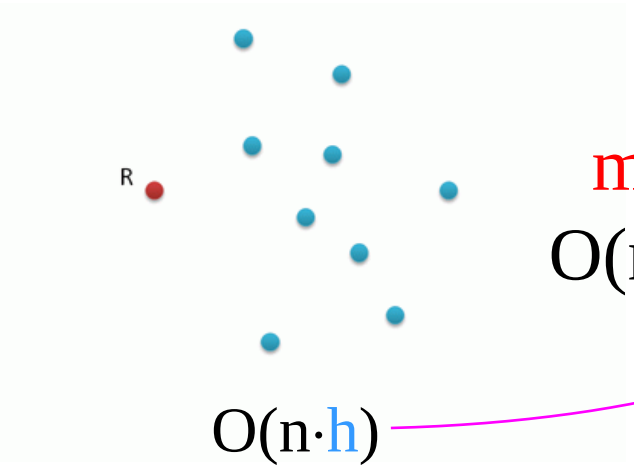
- O(n log n) time to sort
- O(n) time to scan
- O(n log n) time worst-case
- Does not generalize nor parallelizes

Relies on sorting!

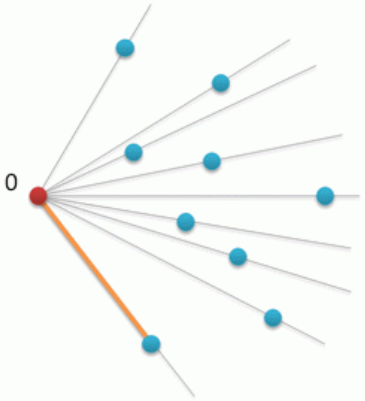


Convex Hulls

Jarvis' march compared to Graham's scan:

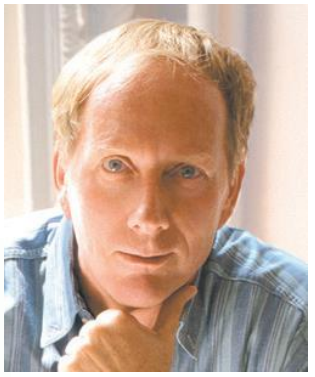


Parallel meta-heuristic: O(n·min(h,log n))



O(n·h)

O(n log n)



Ron Graham

- If $h < \log n \Rightarrow$ Jarvis' march wins
- If $h > \log n \Rightarrow$ Graham's scan wins

Serial meta-heuristic: run Jarvis until $h > \log n$, then run Graham

Expected CH sizes (uniform distributions):

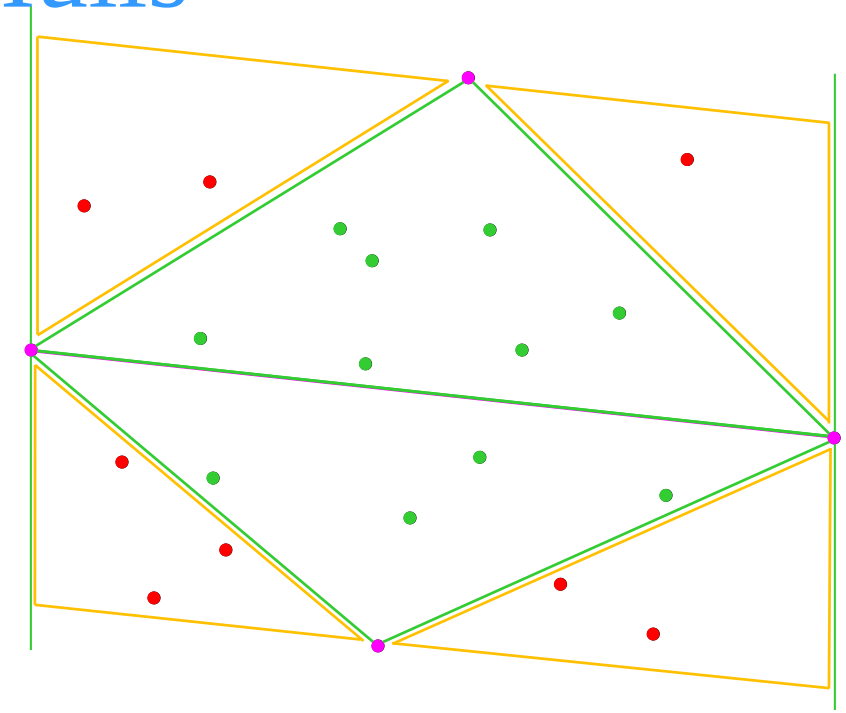
- square: $h = O(\log n)$, r-gon: $h = O(r \cdot \log n)$
- circle: $h = O(n^{1/3})$, sphere: $h = O(n^{1/2})$

E.g., Jarvis' march in a circle: $O(n^{4/3})$

Convex Hulls

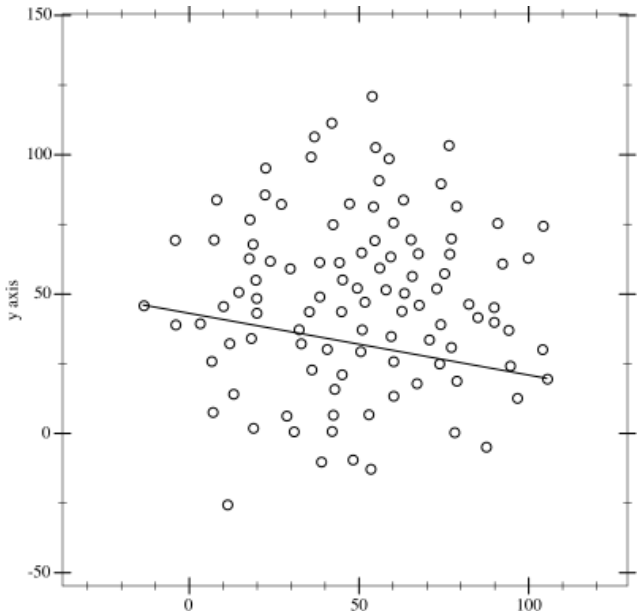
QuickHull: (like QuickSort)

Find right and left –most points
Partition points along this line
Find points farthest from line
Eliminate all internal points
Recurse on outside 4 regions
Concatenate resulting CHs



- O(n log n) expected time
- O(n^2) worst-case time
- Generalizes to higher dim
- Parallelizes

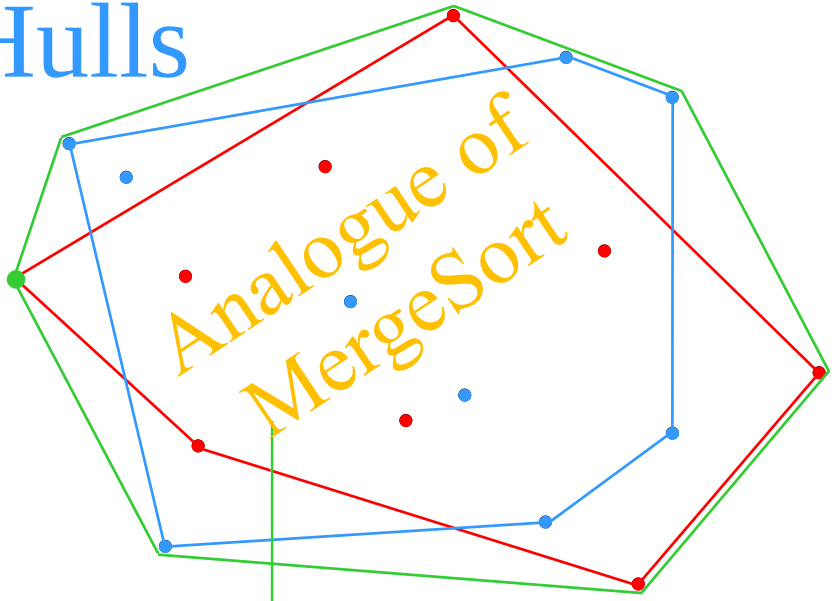
Analogue of QuickSort



Convex Hulls

MergeHull: (like MergeSort)

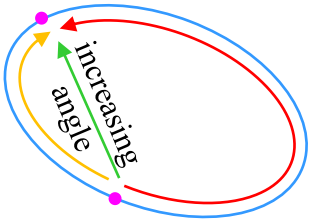
Partition points into two sets
Compute MergeHull of each set
Merge the two resulting CHs



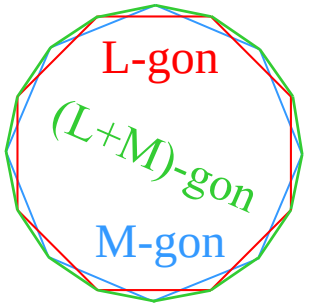
Merging two convex hulls:

Pick point p with least x
Form angle-monotone chains w.r.t. p
Merge chains into angle-sorted list
Run Graham's scan to form CH

Linear time!



CH(L-gon ∪ M-gon)
≤ (L+M)-gon



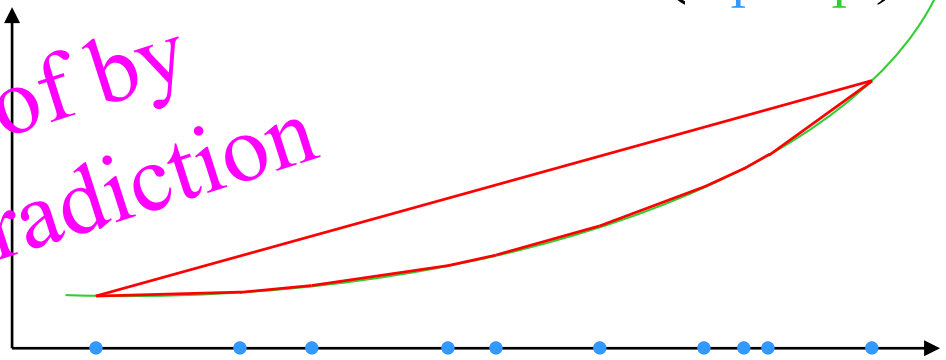
- $T(n) = 2T(n/2) + n = \Theta(n \log n)$
- Generalizes to higher dimensions
- Parallelizes

Lower Bound for Convex Hulls

Theorem: CH requires Ω(n log n) comparisons.

Proof: Reduce sorting to convex hull:

- Consider arbitrary set of numbers x_i to be sorted
- Raise the x_i's to the parabola (x_i, x_i^2)
- Compute convex hull of (x_i, x_i^2)'s



Proof by contradiction

Linear-time transformation of sorting to convex hull

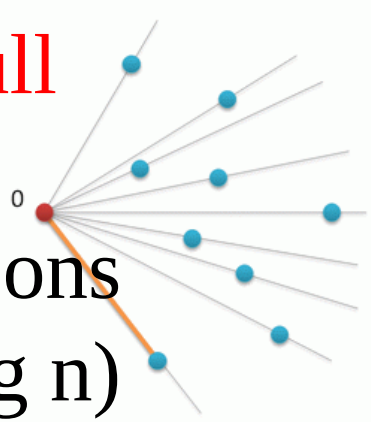
- Sorted order of x_i's is implicit in convex hull

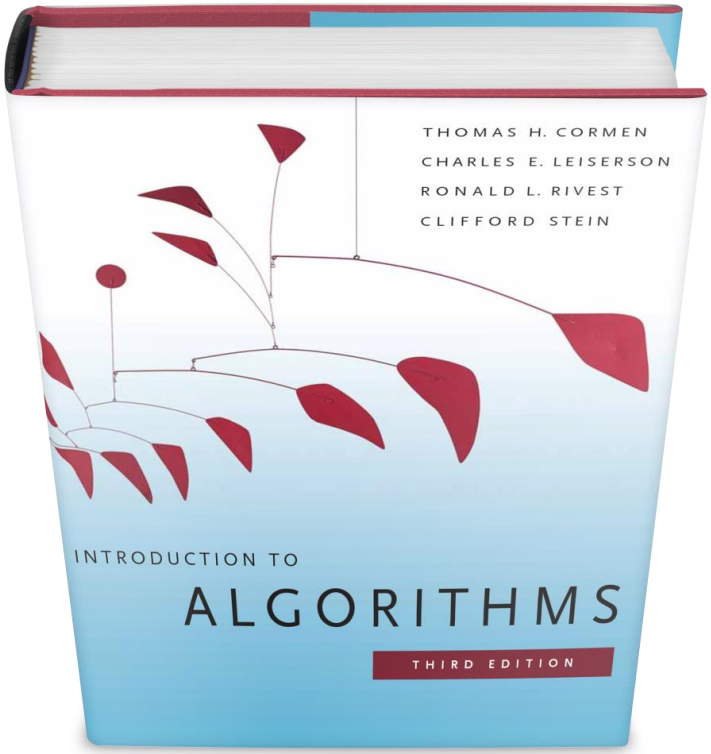
Linear-time!

⇒ Sorting takes no longer than convex hull

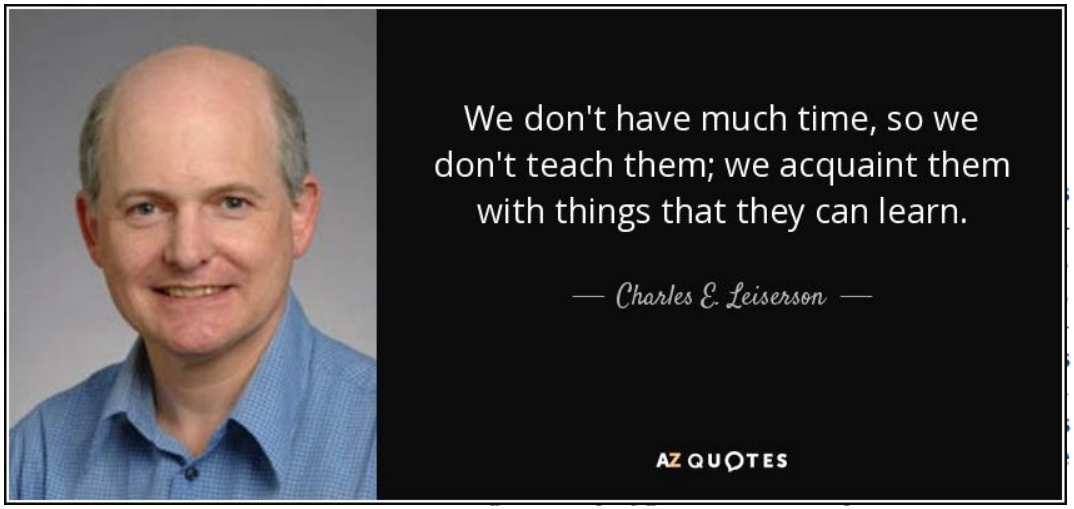
⇒ Convex hull requires Ω(n log n) comparisons

Corollary: Graham's scan is optimal! Θ(n log n)





33 Computational Geometry



In this chapter, we look at a few computational-geometry algorithms in two dimensions, that is, in the plane. We represent each input object by a set of points $\{p_1, p_2, p_3, \ldots\}$, where each $p_i = (x_i, y_i)$ and $x_i, y_i \in \mathbb{R}$. For example, we represent an n-vertex polygon P by a sequence $\langle p_0, p_1, p_2, \ldots, p_{n-1} \rangle$ of its vertices in order of their appearance on the boundary of P. Computational geometry can also apply to three dimensions, and even higher-dimensional spaces, but such problems and their solutions can be very difficult to visualize. Even in two dimensions, however, we can see a good sample of computational-geometry techniques.

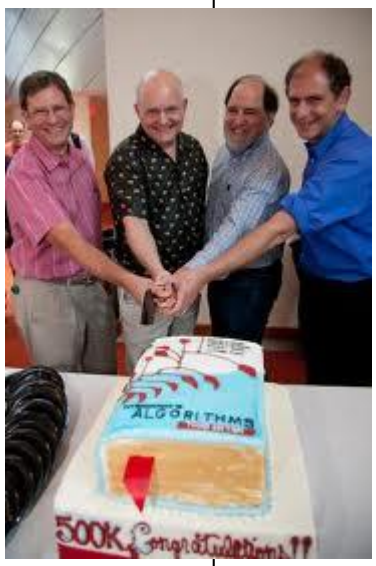
Section 33.1 shows how to answer basic questions about line segments efficiently and accurately: whether one segment is clockwise or counterclockwise from another that shares an endpoint, which way we turn when traversing two adjoining line segments, and whether two line segments intersect. Section 33.2 presents a technique called "sweeping" that we use to develop an $O(n \lg n)$-time algorithm for determining whether a set of n line segments contains any intersections. Section 33.3 gives two "rotational-sweep" algorithms that compute the convex hull (smallest enclosing convex polygon) of a set of n points: Graham's scan, which runs in time $O(n \lg n)$, and Jarvis's march, which takes $O(nh)$ time, where h is the number of vertices of the convex hull. Finally, Section 33.4 gives

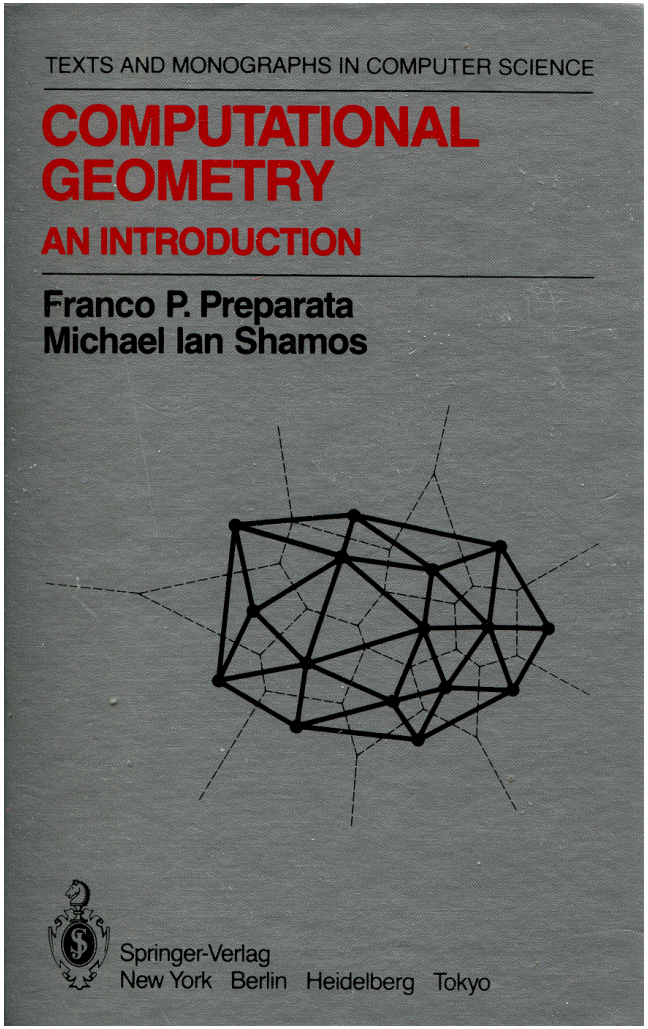

Thomas Cormen


Charles Leiserson


Ronald Rivest


Clifford Stein





Michael Shamos

Franco Preparata

CHAPTER 3

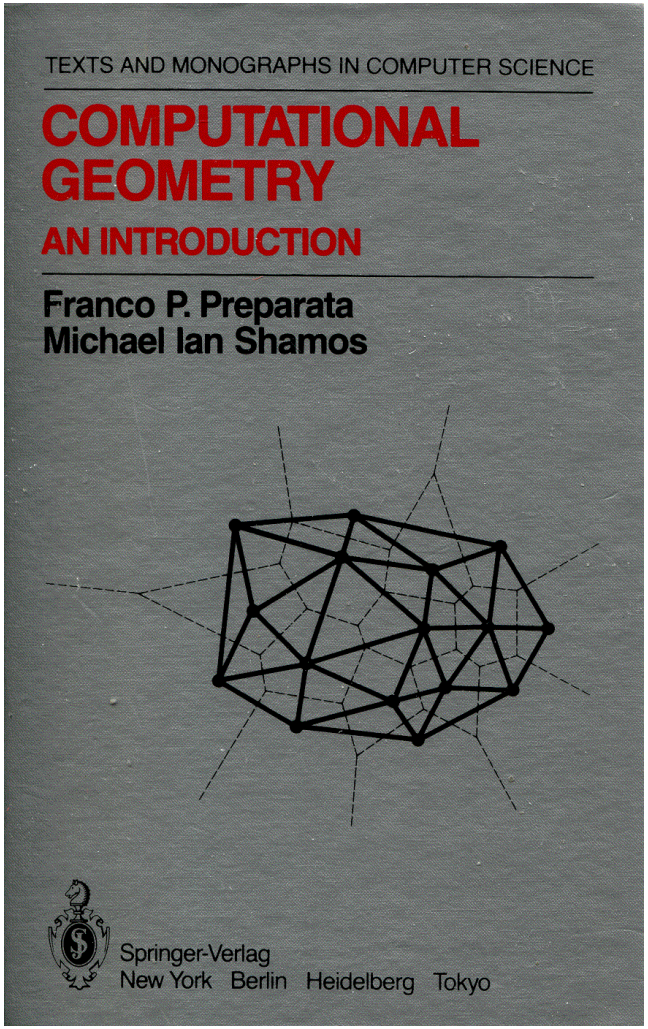
Convex Hulls: Basic Algorithms

The problem of computing a convex hull is not only central to practical applications, but is also a vehicle for the solution of a number of apparently unrelated questions arising in computational geometry. The computation of the convex hull of a finite set of points, particularly in the plane, has been studied extensively and has applications, for example, in pattern recognition [Akl–Toussaint (1978); Duda–Hart (1973)], image processing [Rosenfeld (1969)] and stock cutting and allocation [Freeman (1974); Sklansky (1972); Freeman–Shapira (1975)].

The concept of convex hull of a set of points S is natural and easy to understand. By definition, it is the smallest convex set containing S. Intuitively, if S consists of a finite set of points in the plane, imagine surrounding the set by a large, stretched rubber band; when the band is released it will assume the shape of the convex hull.

In spite of the intuitive appeal of the convex hull concept, the history of algorithms to compute convex hulls is illustrative of a general pattern in algorithmic research. Unfortunately, the simple definition of convex hull recalled above is not of a constructive nature. Thus, appropriate notions must be identified that are conducive to algorithm development.

The construction of the convex hull, in two or more dimensions, is the subject of this chapter; in the next chapter we shall consider applications, variants, and some related—but inherently different—problems. To avoid some repetition, it is convenient to develop a suitable framework for the notions pertaining to convex hulls in arbitrary dimension [Grünbaum (1967); Rockafellar (1970); McMullen–Shephard (1971); Klee (1966)]; these notions have very simple specializations in the ordinary plane and space. The next section is devoted to this task; frequent reference will be made to concepts introduced in Section 1.3.1.




Michael Shamos


Franco Preparata

CHAPTER 4

Convex Hulls: Extensions and Applications

This chapter has two objectives. The first is the discussion of variants and special cases of the convex hull problem, as well as the average-case performance analysis of convex hull algorithms. The second objective is the discussion of applications that use the convex hull. New problems will be formulated and treated as they arise in these applications. Their variety should convince the reader that the hull problem is important both in practice and as a fundamental tool in computational geometry.

4.1 Extensions and Variants

4.1.1 Average-case analysis

Referring to the two-dimensional convex hull algorithms discussed in Section 3.3 of the preceding chapter, we note that Graham's convex hull algorithm always uses $O(N \log N)$ time, regardless of the data, because its first step is to sort the input. Jarvis's algorithm, on the other hand, uses time that varies between linear and quadratic, so it makes sense to ask how much time it can be *expected* to take. The answer to this question will take us into the difficult but fascinating field of stochastic geometry, where we will see some of the difficulties associated with analyzing the average-case performance of geometric algorithms.

Since Jarvis's algorithm runs in $O(hN)$ time, where h is the number of hull vertices, to analyze its average-case performance we need only compute $E(h)$, the expected value of h. In order to do this, we must make some assumption

Point Location

Non-convex case:

Input: polygon and query point

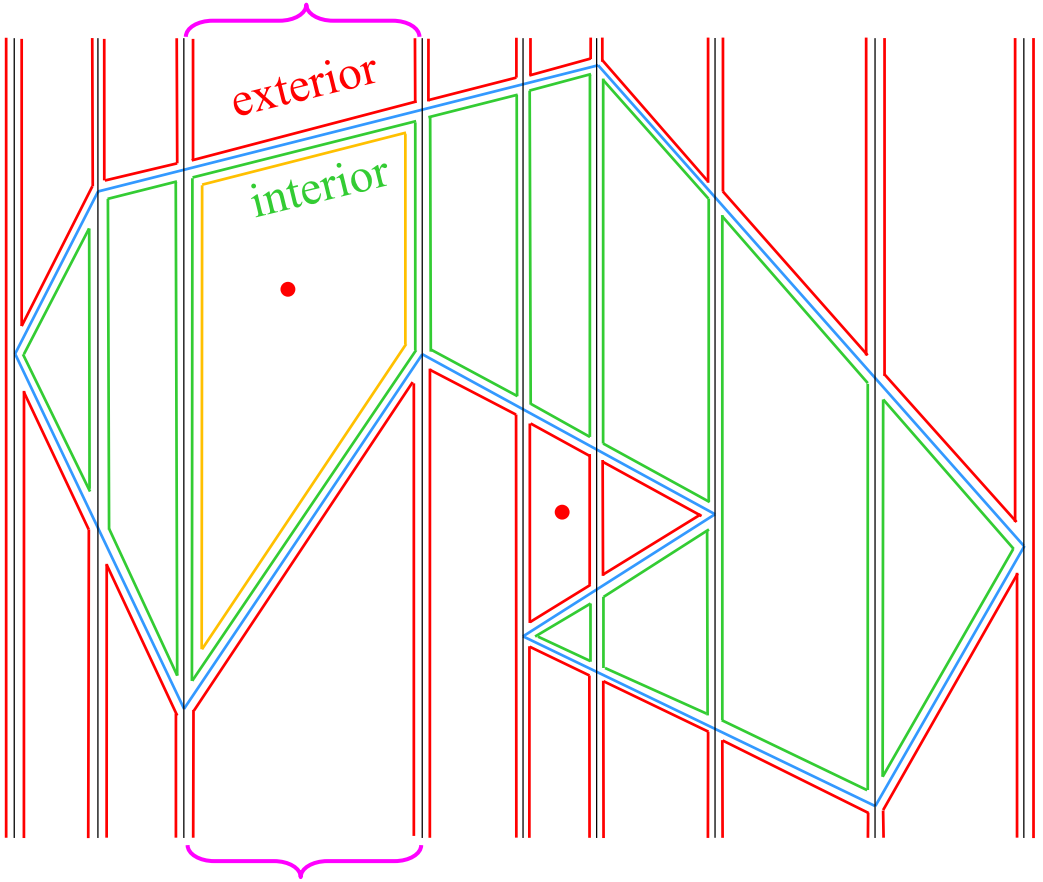
Output: is query point interior to polygon?

Preprocessing:

- Sort vertices by x
- Find vertical slices
- Partition into trapezoids
- Sort slice trapezoids by y

Query:

- Find containing slice
- Find trapezoid in slice
- Report interior/exterior



- $O(\log n)$ time per query (two binary searches)
- $O(n^2)$ space and $O(n^2)$ preprocessing time

Planar Subdivision Search

Arbitrary set of polygons

Input: polygon and query point

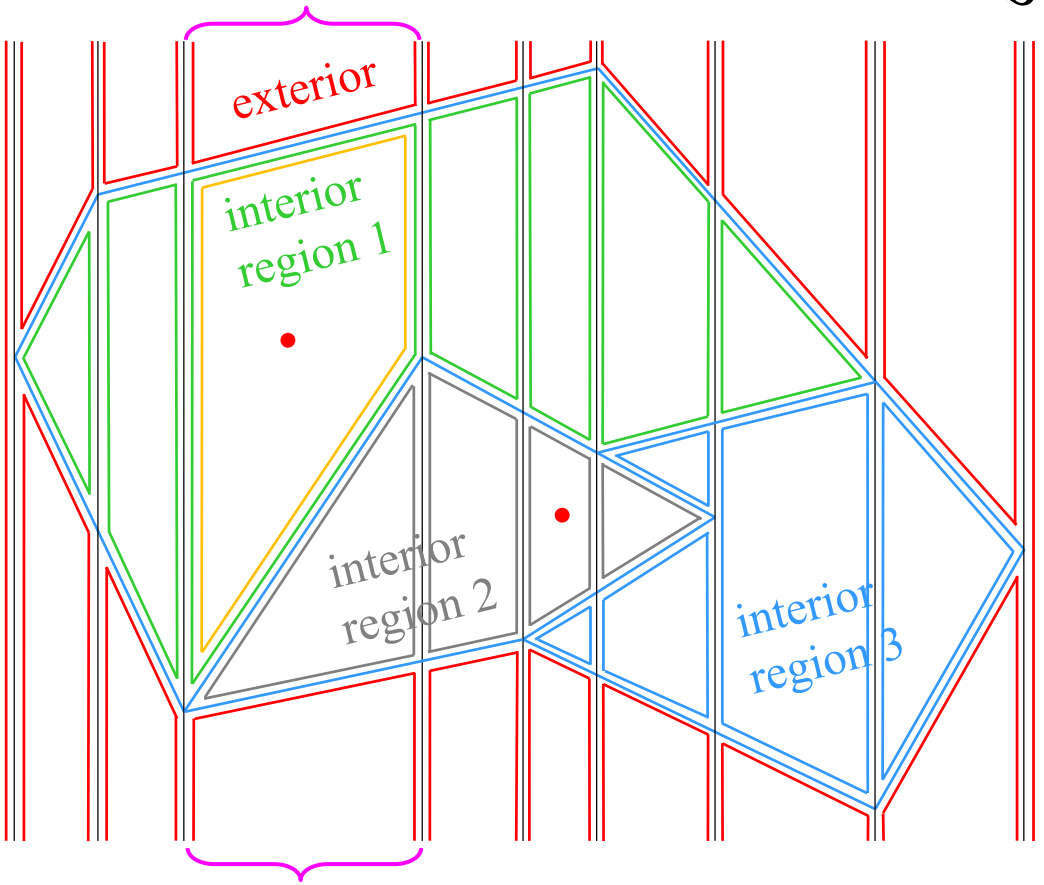
Output: is query point interior to polygon?

Preprocessing:

- Sort vertices by x
- Find vertical slices
- Partition into trapezoids
- Sort slice trapezoids by y

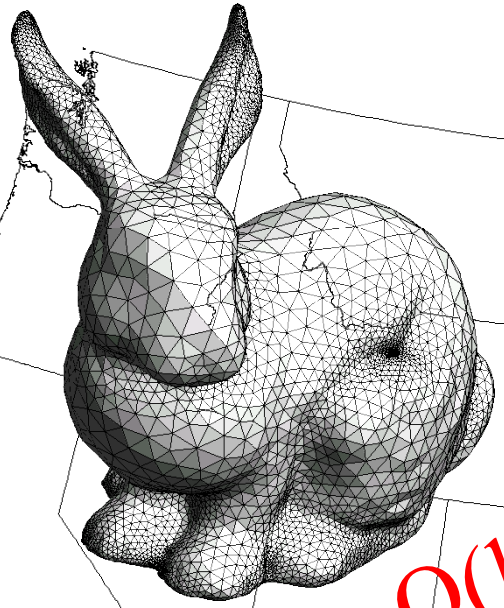
Query:

- Find containing slice
- Find trapezoid in slice
- Report interior/exterior



- O(log n) time per query (two binary searches)
- $O(n^2)$ space and $O(n^2)$ preprocessing time

Planar Subdivision Search



$O(\log n)$ time per query

$O(n^2)$ space, $O(n^2)$ preprocessing, or

$O(n)$ space, $O(n \log n)$ preprocessing

non-contiguous

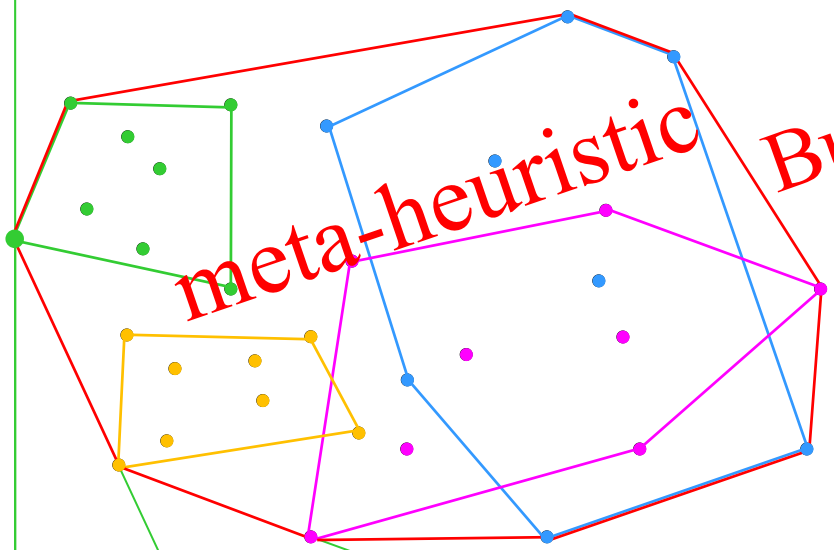
Convex Hulls


Timothy Chan

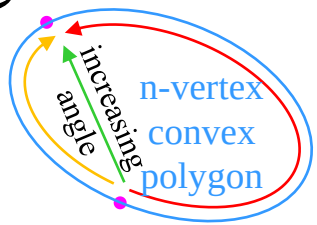
Chan's convex hull algorithm [1996]:

- Assume we know the CH size m=h

Partition points into n/m sets of m each
Compute CH of each set using Graham
Compute m steps of Jarvis on CHs



But h isn't known apriori!
Idea: keep increasing h



Theorem: given a point and n-vertex convex polygon, the tangents can be found in O(log n) time (binary search).

- Time for n/m Grahams: (n/m)·O(m log m) = O(n log m)
- Time for Jarvis: m·(n/m)·O(log m) = O(n log m)
- Total time: O(n log m)

Convex Hulls


Timothy Chan

Chan's convex hull algorithm [1996]:

- $m=h \Rightarrow$ time is $O(n \log h)$
- If m too large $\Rightarrow$ Graham's $O(n \log n)$ dominates
- If m too small $\Rightarrow$ Jarvis' $O(n \cdot h)$ dominates

Q: How can we pick a good m?

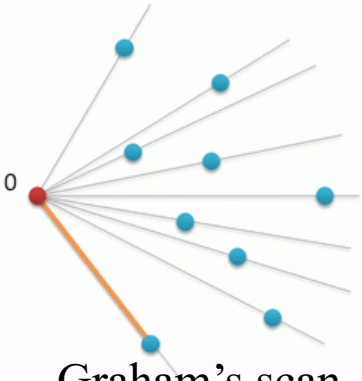
Idea: keep increasing m until m>h

- Start with initial $m=2$
- If m proves too small (<h) square m, abandon (!) previous work & run again
- i.e. $m=2, 2^2, 2^4, 2^8, 2^{16}, \ldots, 2^{2^t} \geq h$ for $t \leq \lceil \log \log h \rceil$
- Time is $O(\sum_{t=1}^{\lceil \log \log h \rceil} n \log 2^{2^t}) = n \cdot O(\sum_{t=1}^{\lceil \log \log h \rceil} 2^t)$
  $= n \cdot O(2^{1+\log \log h}) = O(n \log h)$
- Chan combines two slower algorithms into a faster one!
- Simple, and optimal in both n and h

n/m sets of m each

meta-heuristic

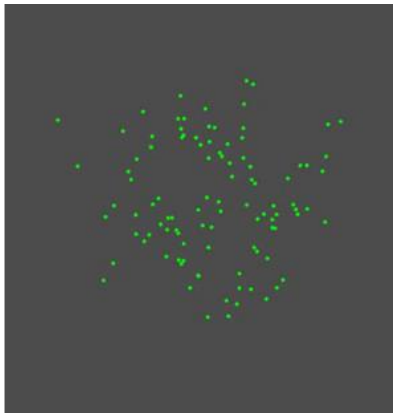
Convex Hulls



Graham's scan

Theorem: The convex hull of a simple polygon can be found in linear time.

Theorem: The convex hull in 3D can be found in optimal time $\Theta(n \log n)$.


3D Jarvis' march

Theorem: The convex hull in d>3 dim can be found in time $\Theta(n^{\lfloor (d+1)/2 \rfloor})$.

Theorem: Identifying the points of the convex hull (unsorted) requires $\Omega(n \log n)$ time (even in 2D).

$\Rightarrow$ $\Omega(n \log n)$ "Hardness" of determining convex hull vertices is sorting-independent

Convex Hulls

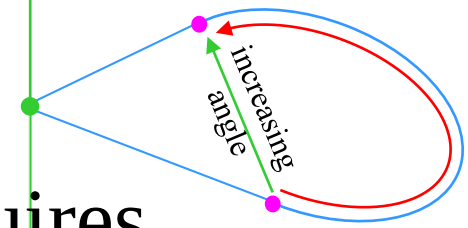
Theorem: Deciding whether all n input points lie on their convex hull requires $\Omega(n \log n)$ time.

Boolean query!

Boolean query!

$\Rightarrow$ Decision "hardness" of convex hull is worse than sorting (deciding "sortedness" is $O(n)$ time)

Theorem: Dynamic CH maintenance doable in $O(\log n)$ time per arriving point.



Theorem: Dynamic CH maintenance requires $\Omega(\log n)$ worst-case time per each arriving point.

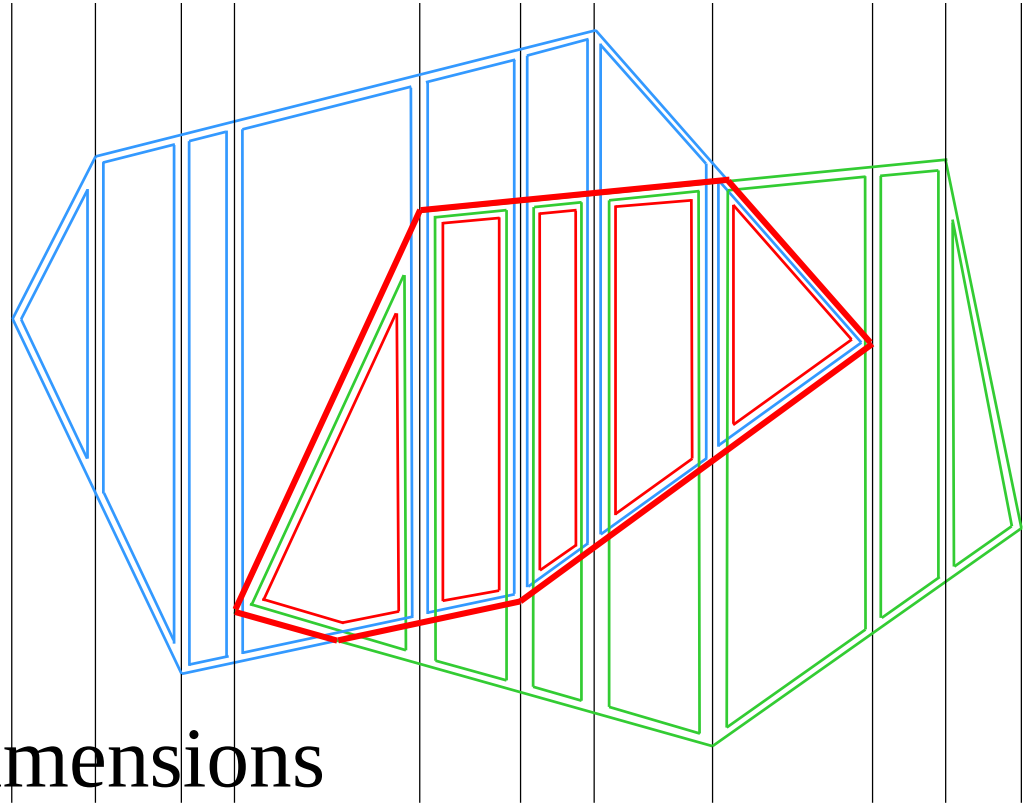
Theorem: Dynamic CH maintenance with deletions can be done within $O(\log^2 n)$ time per point.

Convex Polygon Intersection

Input: two convex polygons

Output: their intersection (polygon)

- Sort both polygons by x
- Find all vertical slices
- Partition into trapezoids
- ∩ trapezoids in each slice
- Stitch ∩'s together



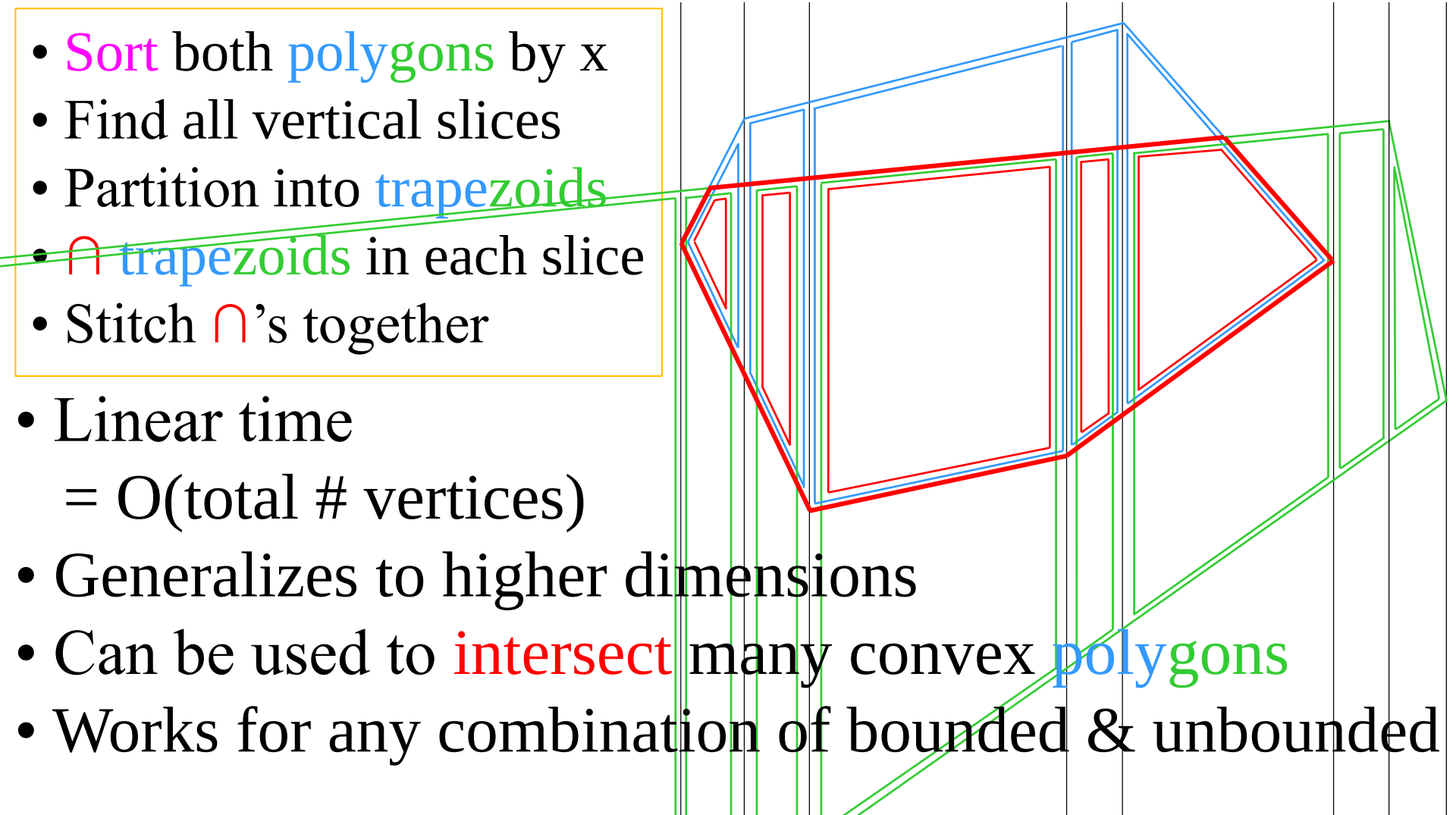
- Linear time
  = O(total # vertices)
- Generalizes to higher dimensions
- Can be used to intersect many convex polygons
- Generalizes to unbounded convex polygonal regions

Convex Polygon Intersection

Input: two convex polygons (possibly unbounded)

Output: their intersection (polygon)

- Sort both polygons by x
- Find all vertical slices
- Partition into trapezoids
- ∩ trapezoids in each slice
- Stitch ∩'s together

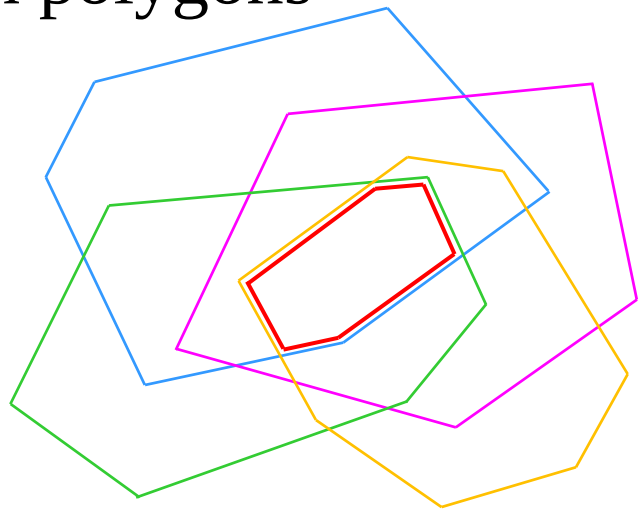


- Linear time
  = O(total # vertices)
- Generalizes to higher dimensions
- Can be used to intersect many convex polygons
- Works for any combination of bounded & unbounded

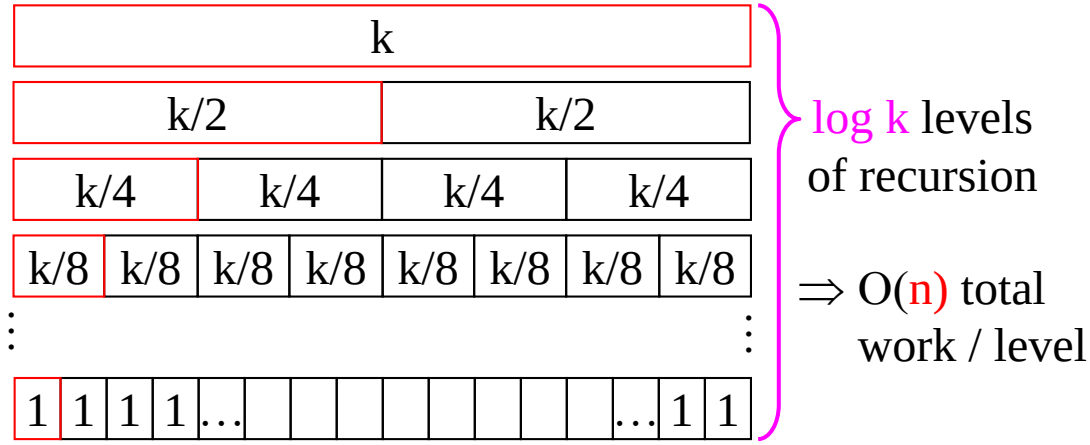
Convex Polygon Intersection

Input: k convex polygons (possibly unbounded)
n total number of vertices of all k<n polygons

Output: their intersection (polygon)



- Recursively intersect:
  polygons 1,..,k/2
  polygons k/2+1,…,k
- Intersect both intersections



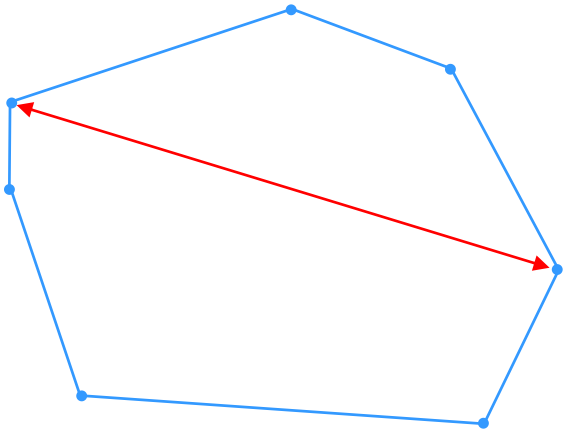
- Total time is Θ(n log k)
- Theorem: Ω(n log k) is necessary for n convex polygon ∩

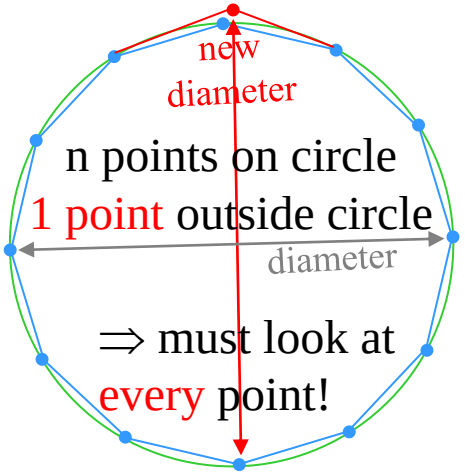
Convex Polygon Diameter

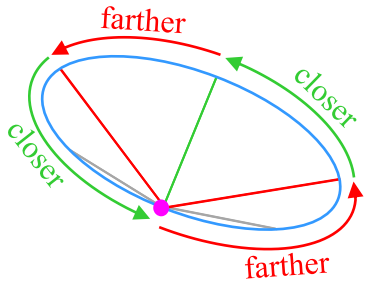
Input: convex polygon with n vertices

Output: diameter (farthest pair)

Theorem: computing the diameter (farthest pair) of a convex polygon requires $\Omega(n)$ time.







"twin peaks"

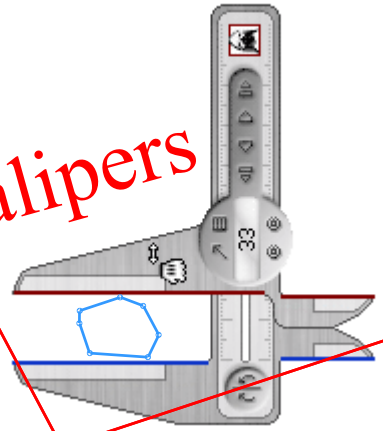
- Naïve algorithm: examine all $\binom{n}{2}$ pairs $\Rightarrow O(n^2)$
- More efficient: for each point, binary-search polygon for farthest other point $\Rightarrow O(n \log n)$

Convex Polygon Diameter

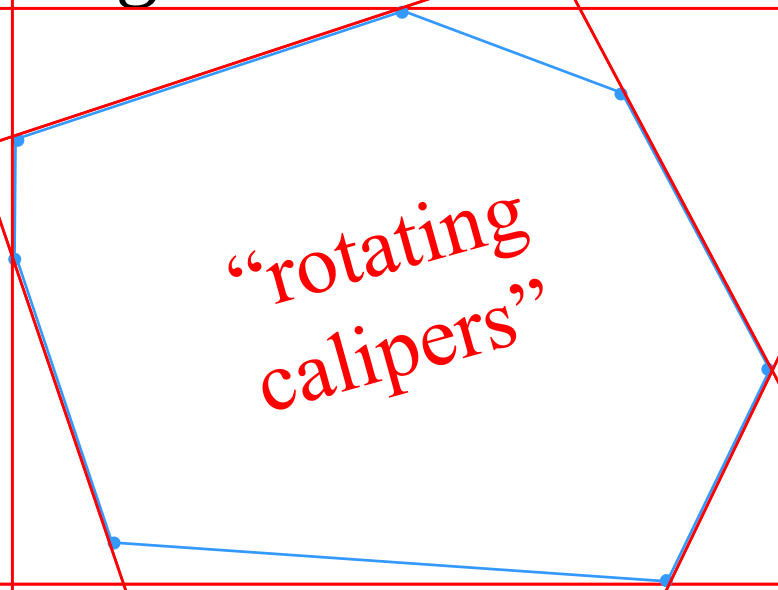
Input: convex polygon with n vertices

Output: diameter (farthest pair)

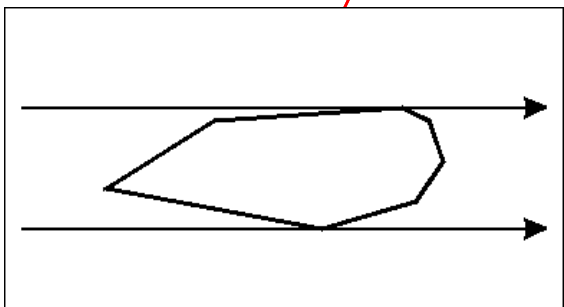
Theorem: Diameter is largest distance between any pair of parallel tangents.



Find horizontal max y tangent
Find horizontal min y tangent
Rotate tangents while parallel
Rotate all the way around
Diameter is max separation



- Linear total time O(n)
- Finds all "antipodal" pairs
- "Rotating calipers" has lots of applications



The *Rotating Calipers*: An Efficient, Multipurpose, Computational Tool

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ABSTRACT

A paper published in 1983 established that the *rotating calipers* paradigm provides an elegant, simple, and yet powerful computational tool for solving several geometric problems. In the present paper the history of this tool is reviewed, and stock is taken of the rich variety of computational two-dimensional problems and applications that have been tackled with it during the past thirty years.

KEYWORDS

Rotating calipers, design and analysis of algorithms, computational geometry, geometric complexity, computer graphics, computer vision, combinatorial optimization, statistics

1 INTRODUCTION

The *rotating calipers* paradigm constitutes a powerful, simple, elegant, and computationally efficient tool that can solve a wide variety of geometric problems in practice. The basic idea first appeared in the 1978 Ph.D. thesis of Michael Shamos, where it was applied to the computation of the maximum distance between the elements of a convex set: the diameter of a convex polygon in the plane [40]. Later I coined the name "Rotating Calipers" for this procedure, and generalized it in a number of ways to solve several other geometric problems. In 1983 I presented some of these results at an IEEE conference in Athens, Greece [46]. Since then the rotating calipers paradigm has been generalized further to solve other problems in two as well as three dimensions. In the present paper the thirty-year history of this tool is reviewed, focusing on the progress made with it on geometry problems and applications, in two-dimensional space.

2 THE ROTATING CALIPERS

The elegant and simple algorithm that appeared in Shamos' thesis [40], showing that the diameter of a convex n-sided polygon may be computed in $O(n)$ time in the worst case, resembles rotating a pair of calipers through 360°, once around the polygon. This concept is illustrated in Fig. 1, which contains a convex polygon, and its two horizontal lines of support (lower and upper) at vertices p_i and p_j, respectively. Note that the lines of support are *directed*, as indicated by their arrows. This permits the specification of the direction of rotation, so that if the lines are rotated in a clockwise direction while being pivoted about vertices p_i and p_j, the angles θ_i and θ_j that the lines make with vertices p_{i+1} and p_{j+1}, will decrease.

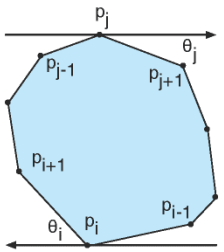

Figure 1. The rotating calipers.

2.1 The Diameter of a Convex Polygon

The *diameter* of a polygon P is the maximum distance between a pair of points in P. Since P contains an infinite number of points, searching all of them is out of the question. To obtain an efficient algorithm we need a characterization of the diameter in terms of a finite subset of the points in P. It is easy to show by a contradiction

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argument, that the diameter of P is determined by a pair of vertices of P. Therefore the diameter may be calculated by examining the distances between all the pairs of vertices of P, and selecting the maximum distance. However, this naïve (brute force) algorithm requires a number of operations that grows as the square of the number of vertices in the polygon. A more fruitful characterization narrows the set of candidates to be searched down to linear size. Several approaches have been tried in the past in order to speed up diameter-finding algorithms [5], and some characterizations have proved to be incorrect [2]. However, a valid characterization was obtained by Shamos [40] via the pairs of vertices, such as p_i and p_j in Fig. 1. These vertices are *antipodal*, meaning that they admit parallel lines of support. The diameter of a polygon is determined by two antipodal vertices. Furthermore, a polygon with n vertices has O(n) antipodal pairs, assuming that all the vertices of P that have an angle of 180° have been removed, which is a straightforward matter. The rotating calipers provide a simple O(n) procedure for searching all the antipodal pairs to find the maximum. The idea is to place a pair of parallel lines of support in any orientation, say horizontal, as in Fig. 1, and then "rotate" the lines, while keeping them as support lines of the polygon, until they are horizontal again. Such a procedure will visit all pairs of antipodal vertices. The crucial observation that makes this seeming infinite continuous process finite is that the rotation can hop from vertex to vertex. Observe in Fig. 1, that as the two lines of support rotate, the vertices p_i and p_j maintain their antipodality property until one of the lines lies flush with an edge of P. In Fig. 1, θ_j is smaller than θ_i, and thus the line advances from p_j to p_{j+1} identifying the next antipodal pair p_i and p_{j+1}. At each step all that is needed is a comparison of two angles to determine which of the two is smaller, which can be done in constant time.

2.2 The Width of a Convex Polygon

The *width* of a polygon P is the minimum distance between a pair of parallel lines of support of P. As with the diameter definition, to obtain an algorithm, the width must be characterized by a finite subset of the lines of support that can be searched efficiently. The following property yields such a characterization. Let p_i and p_j be two vertices of the convex polygon that admit parallel lines of support, and have internal angles less than 180°, as in the example of Fig. 1. If no edge of P lies on the support lines, there exists a preferred direction of rotation for the support lines such that their separation distance decreases. This implies that the width of the polygon is characterized by a vertex and an edge (a vertex-edge pair) that are antipodal, *i.e.*, such that one of the lines of support lies flush with an edge, as shown in Fig. 2. This characterization of the width of a convex polygon found application to the segmentation of plane curves in the work of Ichida and Kiyono [21]. The characterization has also led to several algorithms for computing the width of a polygon. A useful property in this regard is the fact that for a line that contains any edge of a convex polygon P, the perpendicular distance between the line and the vertices of P, as they are traversed in order, defines a *unimodal* function [45]. Kurozumi and Davis [24], and Imai and Iri [22], independently proposed algorithms for computing the width of a convex polygon by visiting each edge of the polygon, and for each edge searching for the vertex furthest from it (in a perpendicular sense). Since this distance function is unimodal the algorithms in [22] and [24] apply binary search to locate these vertices, for each edge of P. This approach results in algorithms with O(n log n) worst-case time complexities.

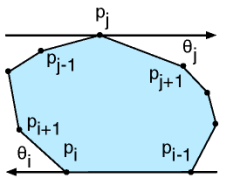

Figure 2. The rotating calipers and the *width* of P.

Houle and Toussaint [20] show that an O(n) time algorithm is achievable by avoiding binary search altogether, and using the rotating calipers instead. To initialize the algorithm, a line of support is constructed through any edge, such as p_{i-1} and p_i in

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Fig. 2, and the vertex furthest from this line is found (p_j in Fig. 2). At each step during the rotation of the lines, the succeeding edge selected is the one that makes the smaller angle with its line of support. This process yields the vertex opposite the edge in only constant time.

2.3 The Minimum-Area Enclosing Rectangle

The algorithm described in the previous sub-section, for computing the width of a polygon with the rotating calipers, represents an application of the original calipers that use two parallel lines of support to compute the diameter, to a different problem. However, the rotating calipers tool itself has also been generalized in several ways. One generalization introduced in [46] uses more than two rotating lines of support. One nice example uses four lines of support to tackle a problem that arises in the areas of image processing and computer vision [11], [39], [50]-[52], optimal packing and layout problems in manufacturing [13], and automatic tariffing in goods traffic [18]. The problem is that of computing the minimum-area enclosing rectangle of a convex n-sided polygon. The usefulness of this rectangle in packing problems is obvious, but it also has applications to shape analysis. For example, the rectangularity of a shape may be measured by the difference in the areas of the shape and its smallest enclosing rectangle [39]. Freeman and Shapira [13] showed that the smallest rectangle must have one of its four edges flush (collinear) with an edge of the polygon, as illustrated in Fig. 3. The algorithm they propose involves visiting every edge of P, such as edge $[p_{i-1}\ p_i]$ in Fig. 3, and locating the three associated extreme vertices that complete the enclosing rectangle, such as p_t, p_j and p_s in Fig. 3. Their algorithm inspects all the vertices of P to find these three extreme vertices, leading to a total computational complexity of $O(n^2)$. If on the other hand each of these extreme vertices is located in O(log n) steps using binary search, instead of a linear scan (which is valid due to the unimodality property of the distances involved [45]), then the solution may be found in O(n log n) time. However, the rotating calipers with four lines of support solves this problem elegantly in O(n) time, as follows. To initialize the procedure any edge of the polygon is selected as the base of a candidate for the smallest enclosing rectangle, and the three extreme vertices are found by a linear scan of all the vertices, as in [13]. The rest of the algorithm proceeds in a manner similar to that used in the algorithm for computing the width of the polygon, except that here the four lines of support are rotated by the smallest of the four angles that the lines make with their succeeding clockwise edges, as shown in Fig. 3. In this way every edge of the polygon generates its candidate rectangle as the support lines make a full revolution around the polygon. Furthermore each candidate rectangle is generated in constant time, resulting in a total time complexity of O(n). An alternate approach to solve this problem in O(n) time, that uses a data structure known as a *star*, is described in [47].

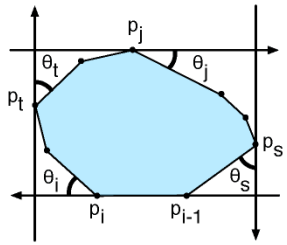

Figure 3. The minimum-area enclosing rectangle.

In closing this sub-section it is worth noting that the rotating calipers can also be used to find minimum-area squares [10], minimum-perimeter enclosures [28], and the densest double-lattice packing of a convex polygon [29].

2.4 The Maximum Distance Between Two Convex Polygons

The maximum distance between two convex polygons, P and Q, arises in several applications including pattern recognition, cluster analysis, and unsupervised learning [12]. It is defined as the largest distance determined by a point in P and a point in Q. As with the diameter of a single polygon, the search for the maximum distance may be restricted to the vertices of P and Q,

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denoted by $p_1, p_2, \ldots, p_n$ and $q_1, q_2, \ldots, q_n$, respectively. This maximum distance between P and Q is given by:

$$d_{max}(P,Q) = \max\{d(p_i, q_j)\},\ i, j=1, 2, \ldots, n, \quad (1)$$

where maximization is done over all i and j, and $d(p_i, p_j)$ is the Euclidean distance between p_i and p_j. Bhattacharya and Toussaint [6] showed that two sets of points may be partitioned into eighteen subsets such that the maximum distance between the two sets is equal to the largest of the eighteen diameters of each of these subsets. Furthermore, the diameter of each subset may be computed with the rotating calipers algorithm applied to their convex hulls. If the two sets are convex polygons, their algorithm runs in O(n) time. However, a simpler and direct O(n) time algorithm for the case of convex polygons was later discovered by Toussaint and McAlear [49]. Their algorithm follows from the generalization of the notion of an antipodal pair of points for a single polygon, as illustrated in Fig. 4.

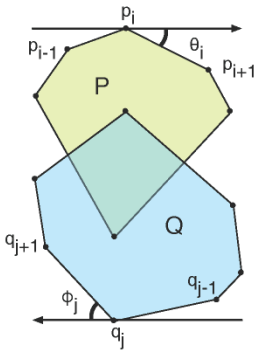

Figure 4. The maximum distance between two convex polygons is determined by an *antipodal pair between* them.

An *antipodal pair between the sets* P and Q, is defined as a pair of vertices $p_i \in P$ and $q_j \in Q$, such that they admit parallel lines of support of P and Q, at p_i and q_j, respectively, with the added restrictions that the support lines are oriented in opposite directions, and each polygon lies to the right of its support line. Note that both polygons need not be contained in the parallel strip defined by the two support lines, as is the case for the configuration in Fig. 4. If one polygon is smaller than the other, or if it lies inside the other, a polygon may protrude outside this strip. It is shown in [49] that the maximum distance between the sets is determined by an antipodal pair between the sets. There are only a linear number of such pairs, and they can be searched in O(n) time by rotating the calipers in the same manner as was done in the diameter algorithm. In Fig. 4, the pair p_i and q_j are one candidate pair, and the next candidate pair is obtained in O(1) time by rotating the calipers in a clockwise manner by the smaller of the angles θ_i and φ_j.

2.5 Minkowski Sum of Two Convex Polygons

Consider two points r and s in the plane, denoted by $r(x_r,y_r)$ and $s(x_s,y_s)$, specified by their x and y coordinates. The Minkowski sum (also called the vector sum) of r and s is the new point $t(x_t,y_t)$, where $x_t = x_r+x_s$ and $y_t = y_r+y_s$. The Minkowski sum of two convex polygons P and Q, denoted by $P \oplus Q$, is the set of points obtained by the Minkowski addition of each and every point in P with each and every point in Q. The Minkowski sums of polygons in the plane, and polyhedra in space, find application in spatial planning problems in the field of robotics [25], [26]. The Minkowski sum of two convex polygons, P and Q may be characterized in terms of their vertices, thus making it computable. In particular, $P \oplus Q$ is a convex polygon, and has at most $2n$ vertices, which are Minkowski sums of the vertices of P with those of Q. This characterization implies the following algorithm: first compute all $\Theta(n^2)$ pairwise Minkowski additions of the vertices of P and Q, and then compute the convex hull of the resulting set. Using an efficient O(n log n) time convex hull algorithm, such as Graham's algorithm [15], yields an O(n^2 log n) time algorithm for computing the Minkowski sum. However, a much faster O(n) time algorithm may be obtained by exploiting a characterization of the Minkowski sum in terms of a modification of the notion of an antipodal pair of vertices. Two vertices $p_i \in P$ and $q_j \in Q$ are defined as being *co-podal* if, and only if, they admit parallel directed

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lines of support of P and Q, at $p_i \in P$ and $q_j \in Q$, respectively, such that the support lines are oriented in the same direction, and each polygon lies to the right of its support line, as illustrated in Fig. 5. The following characterization of the Minkowski sum of P and Q may now be obtained: the vertices of $P \oplus Q$ are the Minkowski sums of *co-podal* pairs of vertices of P and Q. This characterization permits the computation of $P \oplus Q$ by rotating the calipers in the manner as shown in Fig. 5, where the pair p_i and q_j is a candidate pair of vertices to be summed. The subsequent candidate pair is obtained in O(1) time by rotating the calipers in a clockwise manner by the smaller of the angles θ_i and φ_j. Let $z_k = p_i \oplus q_j$ denote a vertex of $P \oplus Q$ that has been computed. Then the succeeding vertex $z_{k+1} = p_i \oplus q_{j+1}$ if $\varphi_j < \theta_i$, $z_{k+1} = p_{i+1} \oplus q_j$ if $\theta_i < \varphi_j$, and $z_{k+1} = p_{i+1} \oplus q_{j+1}$ if $\theta_i = \varphi_j$. Since there are no more than $2n$ vertices in $P \oplus Q$ it follows that O(n) time suffices to compute it.

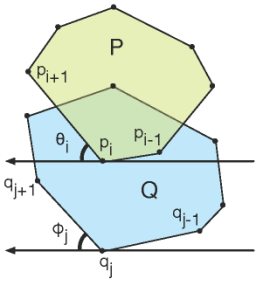

Figure 5. The Minkowski sum of two convex polygons.

2.6 The Convex Hull of Two Convex Polygons

There exist applications where it is required to compute the convex hull of two convex polygons. For example, the divide and conquer approach to computing the convex hull of a set of n points in the plane repeatedly (recursively) merges the convex hulls of two smaller subsets of the points [33]. A linear time merge step is sufficient to yield an algorithm for the convex hull of the set that runs in O(n log n) time. The convex hull of two convex polygons, P and Q, consists of three types of edges: edges of P, edges of Q, and edges that connect vertices of P with those of Q. The latter edges are called *bridges* (also *common tangents*). In Fig. 6 the dashed line L_B that supports polygons P and Q at vertices p_i and q_j, respectively, determines a bridge of the convex hull of P and Q.

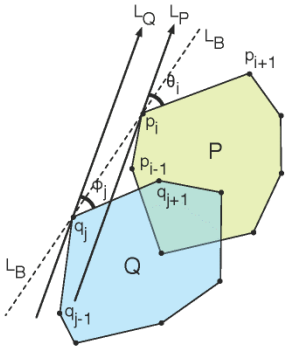

Figure 6. The convex hull of two convex polygons.

The convex hull of two convex polygons may therefore be computed by rotating clockwise two directed lines of support (one on each polygon) oriented in the same direction, such that each polygon is to the right of its line of support, as was done for the Minkowski sum problem. Whenever the two lines of support are not collinear (overlapping) one of them lies to the left of the other. For instance, in Fig. 6, line L_Q lies to the left of L_P. Later in the process line L_P will lie to the left of L_Q. This implies that the lines must, at some time during the rotation, completely overlap, and whenever they do so they identify a bridge. Thus, the convex hull of the two polygons may be obtained by rotating the calipers a full revolution, and at each step outputting the vertex of either P or Q, that lies on the leftmost supporting line. Since each vertex of P and Q is visited only once, and there are O(n) bridges, O(n) time suffices for the entire computation.

The common tangents between two convex polygons have been computed with the rotating calipers in the context of solving special cases of the travelling salesperson problem (TSP) in which there are nested convex obstacles [1].

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2.7 Intersecting Two Convex Polygons

Computing the intersection of two convex polygons is a fundamental operation that occurs in many applications. For example, the divide and conquer approach to computing the intersection of a set of n half-planes, repeatedly merges the intersections of two smaller subsets of the half-planes, which are (perhaps unbounded) convex polygonal sets [34]. A linear-time algorithm for intersecting two convex polygons, that uses the *slab method*, was described by Michael Shamos in his thesis [40], which leads to an O(n log n) time algorithm for the half-plane intersection problem. An alternate O(n) time algorithm was proposed by O'Rourke [31]. A transparently clear O(n) algorithm along with an easy proof of correctness, that uses the rotating calipers was presented in [44]. The algorithm exploits the fact that if two convex polygons intersect, then there exists an intersection point corresponding to each bridge in the convex hull of the two polygons, as illustrated in Fig. 7, where the dashed line L_B identifies a bridge determined by vertices $q_j \in Q$ and $p_i \in P$, and x denotes the intersection point corresponding to this bridge. The algorithm in [44] first determines if the two polygons intersect. If they do then the rotating calipers are used to find the convex hull as described in Sub-section 2.6, after which for each bridge the corresponding intersection point is found by a simple step-down procedure along the convex chains searching for the two intersecting edges.

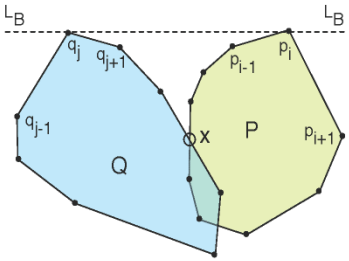

Figure 7. A bridge and its corresponding intersection point x of two intersecting convex polygons.

2.8 The Critical lines of Support of Two Convex Polygons

Given two disjoint convex polygons P and Q, the critical support lines are the two lines that separate P and Q, such that they are both support lines for P and Q. In Fig. 8 the two critical support lines, L_P and L_Q, are the dashed lines. One line contains vertices p_{i+1} and q_j, and the other contains p_{i-1} and q_{j-1}. Intuitively, the two critical support lines may be obtained by rotating any line that separates the two polygons, in clockwise and counterclockwise directions as much as possible, while maintaining the separability of the two polygons. Critical support lines find application to a variety of problems, some of which are described in the following subsections. For two convex polygons they can be computed in linear time using the rotating calipers in a manner similar to that of computing the maximum distance, by initially placing the two parallel lines oriented in opposite directions such that each polygon is to the right of its support line, as in Fig. 4. During the rotation phase of the calipers the critical support lines are detected each time that both support lines are collinear (overlap each other), as in Fig. 8.

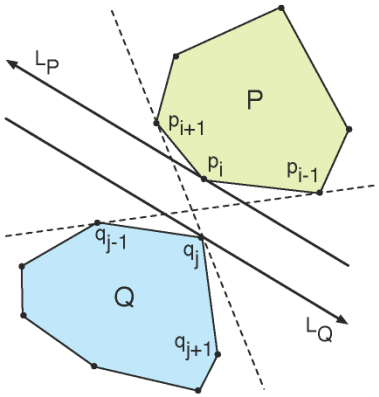

Figure 8. The two critical support lines (dashed) determined by two disjoint convex polygons.

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2.9 The Widest Separating Strip Between Two Convex Polygons, and Machine Learning

In the context of pattern classification and machine learning, two polygons P and Q may be thought of as regions in a feature space that enclose points of the training data for a two-class discrimination problem. Any separating line then is a linear classification rule that can be used to classify future patterns (points) depending on whether they lie on one side or the other of this line. In order for the classifier to make more confident decisions it is desired to pick the separating line that is furthest from the two polygons. Such a line may be chosen as the centerline of the widest empty strip that separates the polygons. Such classifiers are referred to as large-margin classifiers [14], (also wide separation of sets [19]) and make up the geometric backbone of support vector machines [27]. For the case of two planar convex polygons the widest empty strip is determined by either one vertex from each polygon, or by a vertex of one polygon and an edge of the other. Furthermore, this strip may be detected when the support lines are oriented in between the directions of the two critical support lines, and may be found in O(n) time with the rotating calipers.

The widest empty strip problem is closely related to the problems of fitting lines to data [36], finding transversals of sets [4], and linear approximation of objects [37], [38], all of which have been solved efficiently using the rotating calipers.

2.10 The Grenander Distance Between Two Convex Polygons

Ulf Grenander [16] proposed that the distance between two convex polygons be measured by comparing the lengths of the connecting segments of their critical support lines, to the lengths of the polygonal chains spanned by these segments. To be more precise let the critical support lines be supported at p_i and p_r in P, and q_j and q_s in Q, and refer to Fig. 9. The supporting vertices of the critical support lines partition the convex polygons into two polygonal chains: the *inner chains* facing the intersection point of the support lines, and the complementary *outer chains*. The *supporting chords* of P and Q are the line segments that connect q_j to p_i and p_r to q_s, shown as bold red line segments in Fig. 9. The Grenander distance is defined as the sum of the lengths of the supporting chords, less the sum of the lengths of all the edges of the polygons that belong to the inner chains. Clearly, once the critical support lines have been computed with the rotating calipers, O(n) time suffices for the computation of the lengths of the chords and chains, and therefore the Grenander distance may be computed in O(n) time.

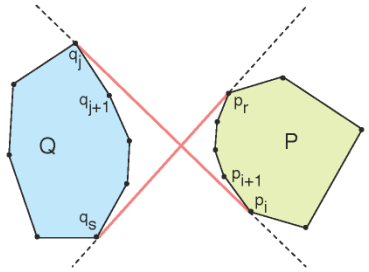

Figure 9. The Grenander distance between two convex polygons.

2.11 Optimal Strip Separation in Medical Imaging and Solid Modeling

In several contexts such as medical imaging it is required to construct a solid model by stitching parallel polygonal slices together. A problem arises during the interpolation when the solid object is bounded by a single contour in one slice, and two contours in the adjoining slice. The computational geometric problem that results is the following [3]. Given two linearly separable polygons P and Q, and a third convex polygon R, it is required to compute the separating strip between P and Q, that covers the largest area of R. Barequet and Wolfers [3] present a linear-time algorithm for computing this optimum strip using the rotating calipers. They also consider the case when the polygon R is not convex, but in this case the running time of their algorithm is quadratic in the size of the input.

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2.12 Aperture Angle Optimization for Visibility Problems in Graphics and Computer Vision

In several disciplines such as computer graphics, computer vision, robotics, operations research, visual inspection, and accessibility analysis in the manufacturing industry, the notion of visibility is fundamental. Often models assume that a camera can see in all directions, in effect idealizing the camera's aperture angle to 360°. In a more realistic model the aperture angle is much smaller than 360°. Furthermore, if the camera is movable, it is desirable to compute the maximum and minimum aperture angles that the camera may need as it travels in a constrained space. Let P and Q be two disjoint convex polygons in the plane. For a given a point x in P, the aperture angle at x with respect to Q is defined as the angle of the cone with apex at x, that contains Q, and has its two rays that emanate from x tangent to Q. The critical lines of support play singular roles in computing the extreme aperture angles, and may be efficiently computed with the rotating calipers [7].

2.13 Wedge Placement Optimization Problems

Wedge placement optimization problems arise in several contexts such as visibility with bounded aperture angles, and layout design of parts in stock cutting for manufacturing. A wedge W may be thought of simply as an unbounded cone with a fixed angle θ at its apex. Given an n-vertex polygonal region, such as R in Fig. 10, we are interested in computing the entire region where a camera (the apex of the cone) with aperture angle θ, may be positioned so that it is as close as possible to R. Such a region is bounded by a concatenation of arcs (called the *wedge cloud*) determined by the apex of the cone as it travels around R maintaining contact with R. Fig. 10 shows three points on the cloud, x_i, x_j and x_k which are camera locations with a fixed aperture angle θ. The original rotating calipers can be generalized so that the lines of support form any fixed angle to each other. Teichmann [41] used such a generalization of the calipers to compute the wedge cloud of a convex polygon in O(n) worst-case time. Note that to maintain contact with the polygon R, a wedge must both rotate and translate.

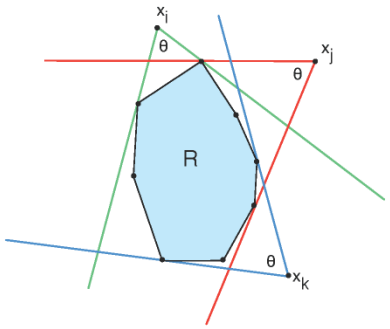

Figure 10. Wedge fitting and the *wedge cloud*.

2.14 Nonparametric Decision Rules

In the nonparametric discrimination problem we are given training data that belong to different classes, and it is desired to classify new incoming data into their respective classes. Jean-Paul Rasson and Grandville [35] proposed a geometric approach to the design of such a decision rule. Consider the two-dimensional, two-class problem, in which the classes are linearly separable, and refer to Fig. 11. As part of the training phase of the classifier, the convex hulls are computed and stored for each class. These convex hulls are the convex polygons P and Q in Fig. 11, colored light green and light blue, respectively. The decision rule for a new incoming pattern X is as follows. If X lies in P (or Q) it is classified as belonging to class P (or Q). If X lies outside both polygons it is classified to the class associated with the nearest polygon, where nearest is defined in terms of the area distance between X and each of the polygons. More precisely, the distance between X and Q is the absolute value of the difference between the area of polygon Q and the area of the convex hull of $Q \cup X$. Similarly, the distance between X and P is the absolute value of the difference between the area of polygon P and the area of the convex hull of $P \cup X$. These areas are colored dark blue and dark green, respectively. In this example the dark green area is smaller than the dark blue area, and therefore X would classified as belonging to class P. The four lines that connect X to the polygons are critical support lines between X and the

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polygons, where X may be considered as a degenerate single-vertex polygon, and may be computed in linear time with the rotating calipers.

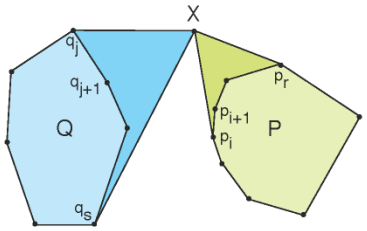

Figure 11. A geometric non-parametric decision rule.

2.15 Nice Triangulations and Quadrangulations of Planar Sets of Points

A convex polygonal *annulus* is the region in between two properly nested convex polygons, such as the blue shaded region consisting of the polygon Q less the interior of the green shaded polygon P shown in Fig. 12. This region admits a very simple triangulation by means of the rotating calipers [43]. A triangulation of a polygonal region is a partition of the region's interior into as many triangles as possible, obtained by inserting interior edges (diagonals) only between pairs of vertices, without allowing any edges to cross each other. For the annulus the calipers work in a similar manner as in the convex hull problem. Initially two parallel lines of support, oriented in the same direction, are constructed through the vertices of P and Q with lowest y-coordinates, so that the polygons lie to the right of the lines (Fig. 12). As the calipers are rotated clockwise, the pairs of vertices that come into contact with the lines of support are connected with an edge. The first few edges connected by this algorithm are shown in red. In general a region admits many triangulations, some of which have long edges, very acute triangles, or other properties deemed undesirable for some applications. One attractive property of the triangulation of the annulus obtained with the rotating calipers algorithm is that the triangulation tends to be nice, in the sense that the resulting triangles tend to be nearly regular. Furthermore, the method can be applied to more general problems such as obtaining nice triangulations of sets of points. One way of doing this is to first compute all the convex layers of the set [9], which yields a nested collection of annuli, each of which can be triangulated with the rotating calipers [42]. Another method involves first computing a spiral polygonal chain spanning the points [23], and then triangulating this spiral with the rotating calipers [42]. This latter triangulation also has the added nice property that it is *serpentine*, *i.e.*, its dual graph is a chain.

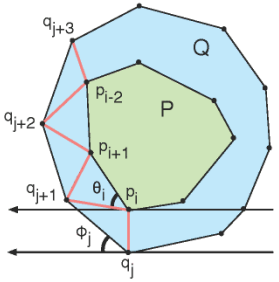

Figure 12. Triangulating a polygonal convex annulus.

3 CONCLUSION

This paper has focused on extensions of the rotating calipers, and their applications to a variety of geometric problems in the plane. The problem of computing the *minimum* distance between sets is conspicuously absent. This is because it appears difficult to crack it with the rotating calipers [48]. The rotating calipers have also been generalized to work on surfaces such as spheres and cones [17], and three-dimensional space, where supporting planes, rather than supporting lines, are rotated [8], [30]. However, a survey of this topic is beyond the scope of this paper, but will be forthcoming in the near future. Finally, the reader is referred to the web page and thesis of Hormoz Pirzadeh [32], for details of some proofs, and animation applets that help to visualize several algorithms that have been described in this paper. http://cgm.cs.mcgill.ca/~orm/welcome.html

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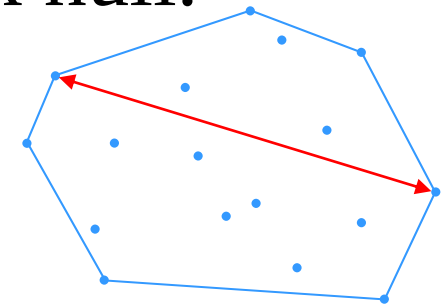
Pointset Diameter

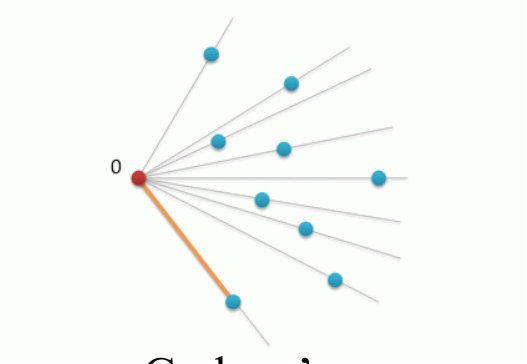
Theorem: The diameter (farthest pair) of a pointset is equal to the diameter of its convex hull.



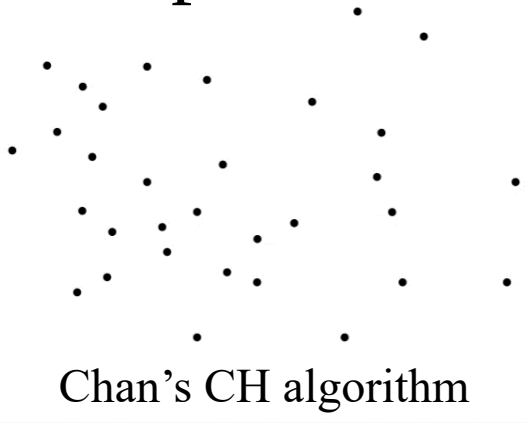
Theorem: Finding the diameter of a pointset requires time $\Omega(n \log n)$.

Theorem: The diameter of a planar pointset can be found in optimal time $\Theta(n \log n)$.

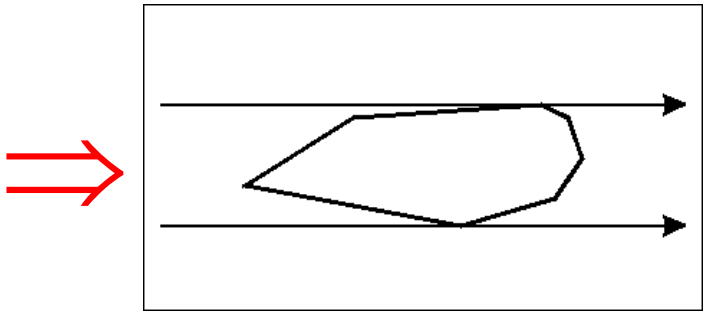
Theorem: The diameter of a planar pointset can be found in optimal time $\Theta(n \log h)$.



Graham's scan

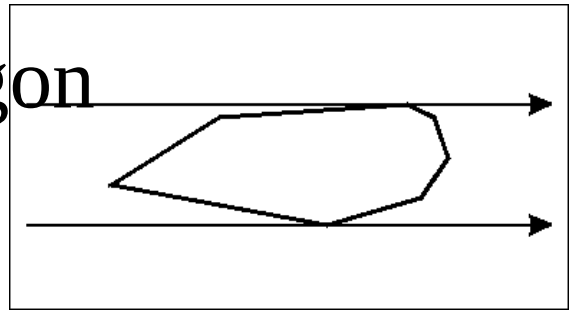


Chan's CH algorithm



Rotating calipers

Pointset Diameter



Theorem: The width of a convex polygon can be found in linear time.

Theorem: The diameter of a simple polygon can be found in linear time.

Theorem: Element uniqueness (whether any pair is equal) requires $\Omega(n \log n)$ time to solve.

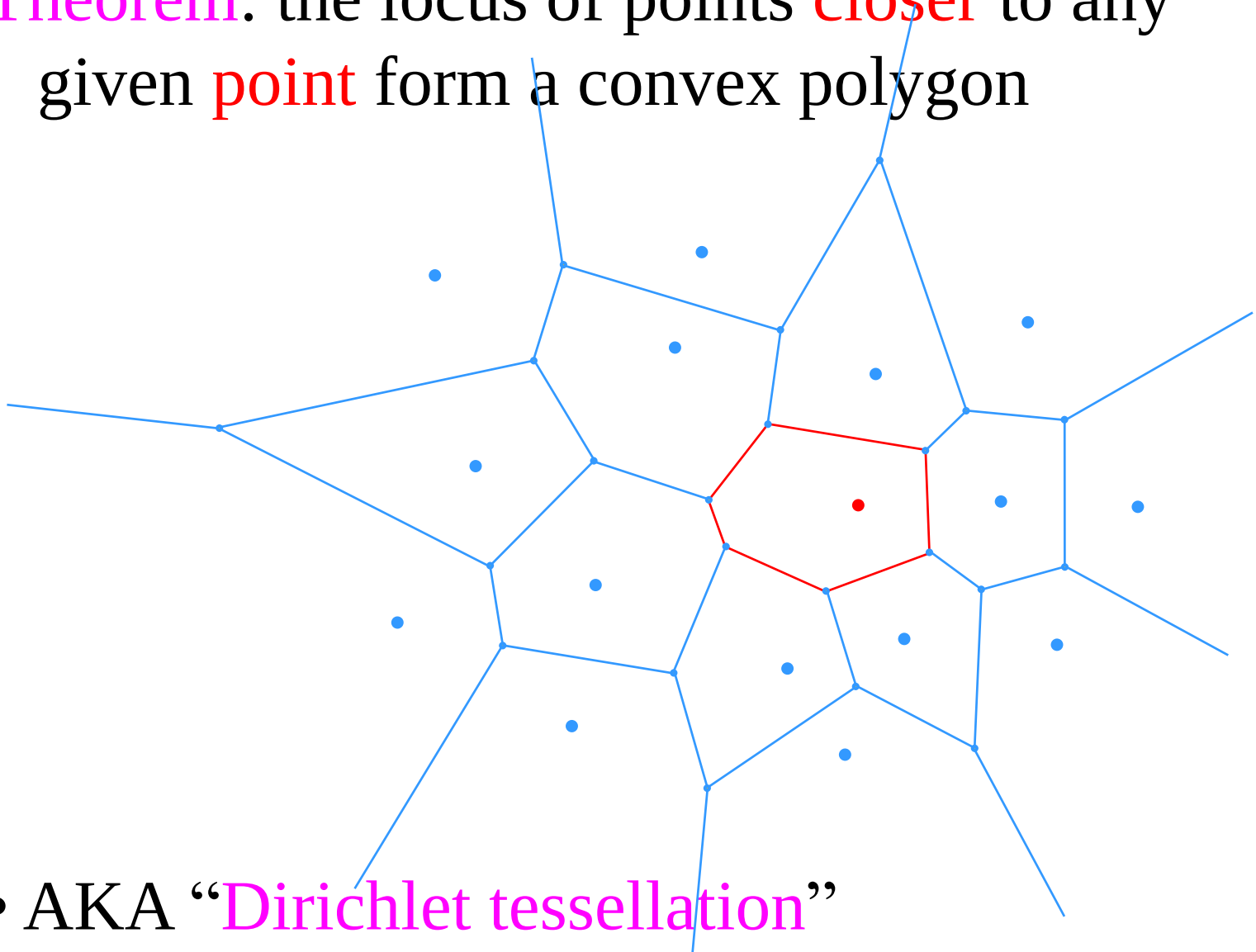
Theorem: Element uniqueness (whether any pair is equal) can be solved in time $\Theta(n \log n)$.

Sorting!

Voronoi Diagrams

Input: set of n points

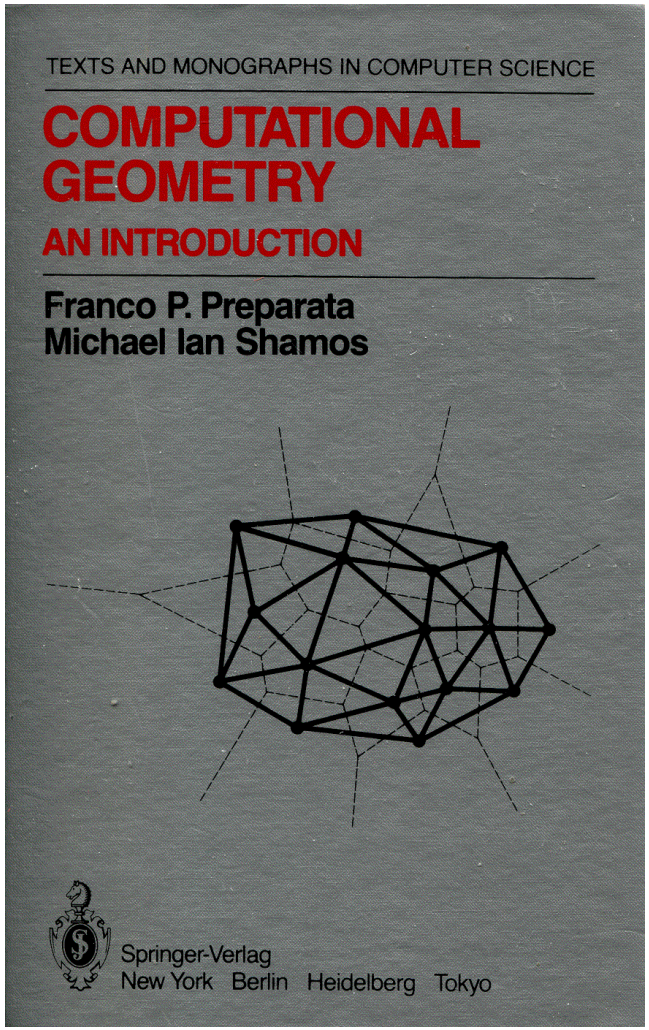
Theorem: the locus of points closer to any given point form a convex polygon



Peter Dirichlet
1805-1859

Georgy Voronoi
1868-1908

- AKA "Dirichlet tessellation"




Michael Shamos


Franco Preparata

Chapter 5
Section 5.5

rence relation (5.2) has solution $P(N, d) = O(N \log N)$. This computational work is of the same order as that of the initial presorting. So we have

Theorem 5.6. *The determination of a closest pair of points in a set of N points in E^d can be completed in time $\theta(N \log N)$, and this is optimal.*

5.5 The Locus Approach to Proximity Problems: The Voronoi Diagram

While the previous divide-and-conquer approach for the closest-pair problem is quite encouraging, it even fails to solve the ALL NEAREST NEIGHBORS problem, which would seem to be a simple extension. Indeed, if we try to set up the analogous recursion for ALL NEAREST NEIGHBORS, we find that the natural way of splitting the problem does not induce sparsity, and there is no apparent way of accomplishing the merge step in less than quadratic time. On the other hand, a valuable heuristic for designing geometric algorithms is to look at the defining loci and try to organize them into a data structure. In a two-dimensional formulation, we want to solve

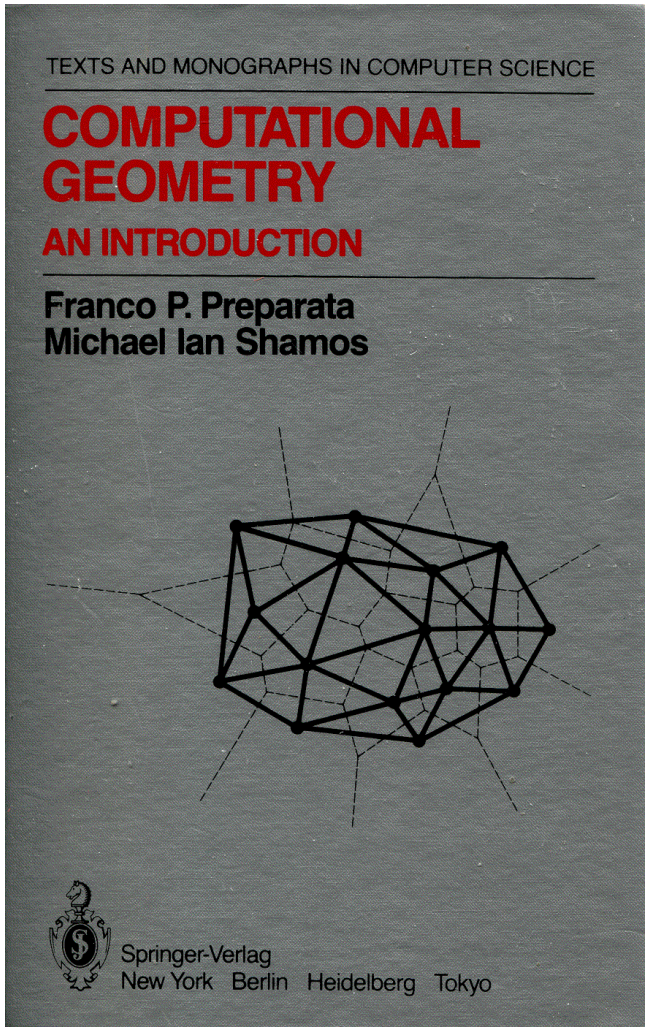
PROBLEM P.8 (LOCI OF PROXIMITY). Given a set S of N points in the plane, for each point p_i in S what is the locus of points (x, y) in the plane that are closer to p_i than to any other point of S?

Note that, intuitively, the solution of the above problem is a partition of the plane into regions (each region being the locus of the points (x, y) closer to a point of S than to any other point of S). We also note that, if we know this partition, by searching it (i.e., by locating a query point q in a region of this partition), we could directly solve the NEAREST-NEIGHBOR SEARCH (Problem P.5). We shall now analyze the structure of this partition of the plane. Given two points, p_i and p_j, the set of points closer to p_i than to p_j is just the half-plane containing p_i that is defined by the perpendicular bisector of $\overline{p_i p_j}$. Let us denote this half-plane by $H(p_i, p_j)$. The locus of points closer to p_i than to any other point, which we denote by $V(i)$, is the *intersection of* $N - 1$ *half-planes*, and is a convex polygonal region (see Section 1.3.1) having no more than $N - 1$ sides, that is,

$$V(i) = \bigcap_{i \neq j} H(p_i, p_j).$$

$V(i)$ is called the *Voronoi polygon associated with* p_i. A Voronoi polygon is shown in Figure 5.16(a) [Rogers (1964)].[12]

[12] These polygons were first studied seriously by the emigré Russian mathematician G. Voronoi, who used them in a treatise on quadratic forms [Voronoi (1908)]. They are also called Dirichlet regions, Thiessen polygons, or Wigner–Seitz cells. Dan Hoey has suggested the more descriptive (and impartial) term "proximal polygons."



Chapter 5
Section 5.5


Michael Shamos


Franco Preparata

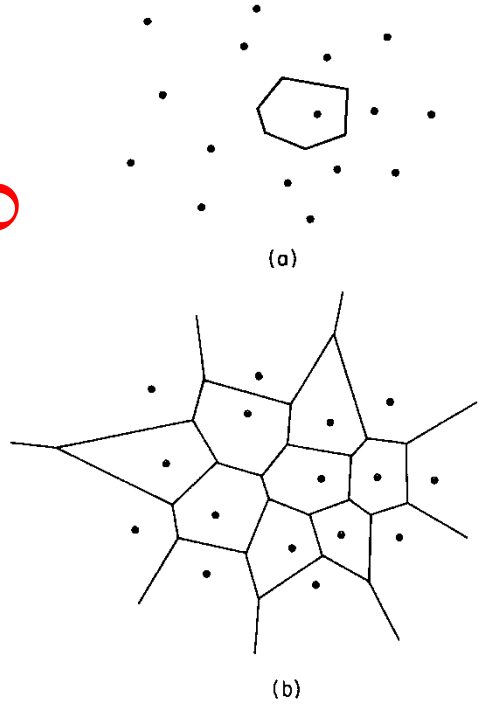


Figure 5.16 (a) A Voronoi polygon; (b) the Voronoi diagram.

These N regions partition the plane into a convex net which we shall refer to as the *Voronoi diagram*, denoted as Vor(S), which is shown in Figure 5.16(b). The vertices of the diagram are *Voronoi vertices*, and its line segments are *Voronoi edges*.

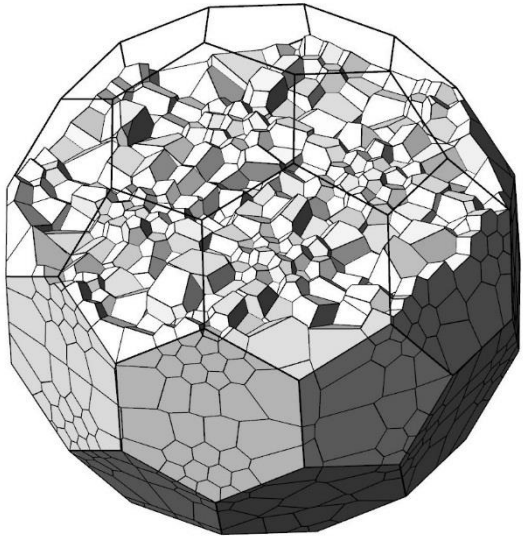
Each of the original N points belongs to a unique Voronoi polygon; thus if $(x, y) \in V(i)$, then p_i is a nearest neighbor of (x, y). The Voronoi diagram contains, in a powerful sense, all of the proximity information defined by the given set.

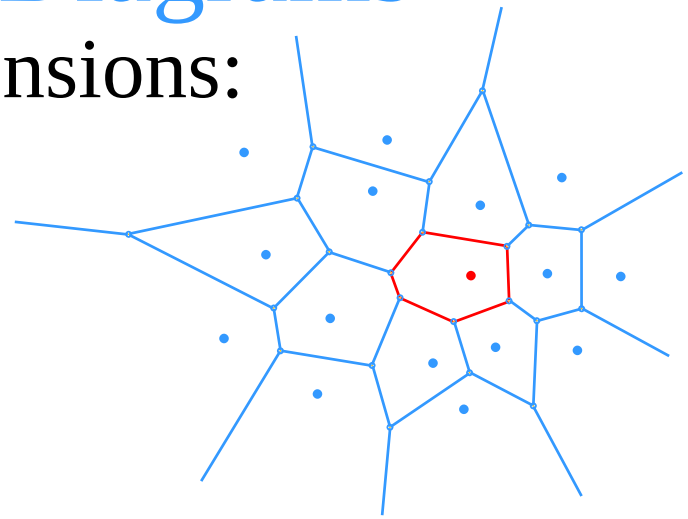
5.5.1 A catalog of Voronoi properties

In this section we list a number of important properties of the Voronoi diagram. Although the Voronoi diagram can be defined for any number of dimensions, our review will refer to the planar case for a two-fold reason: first, to maintain an immediate link with intuitive evidence; second, to focus the

Voronoi Diagrams

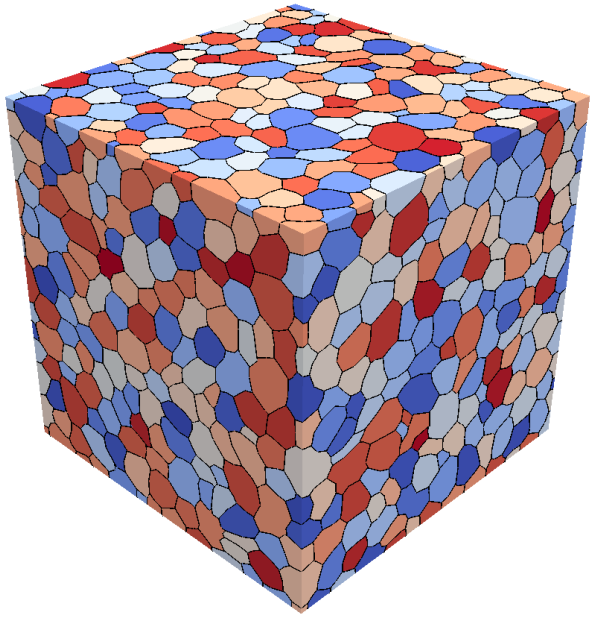
Generalizes to higher dimensions:

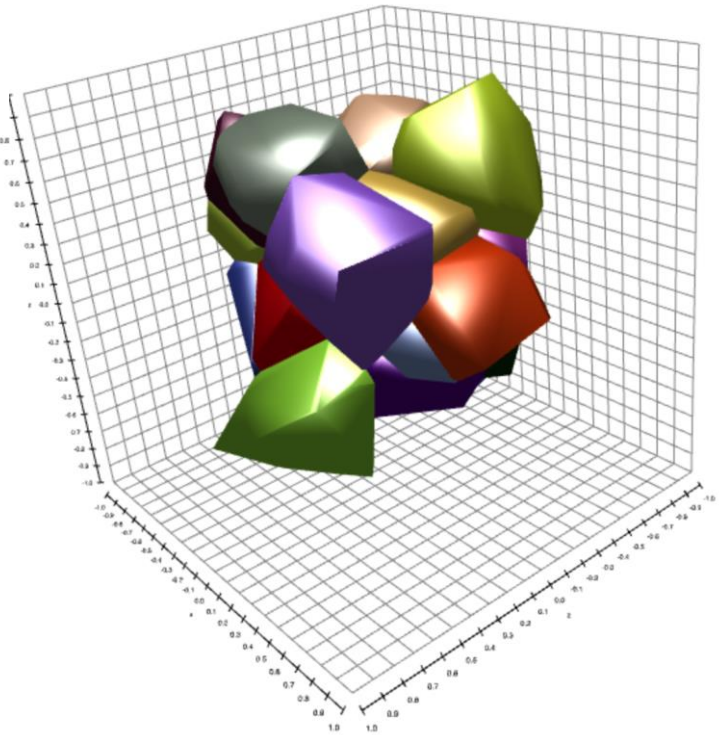




Peter Dirichlet
1805-1859

Georgy Voronoi
1868-1908





Voronoi Diagrams

Generalizes to other metrics:

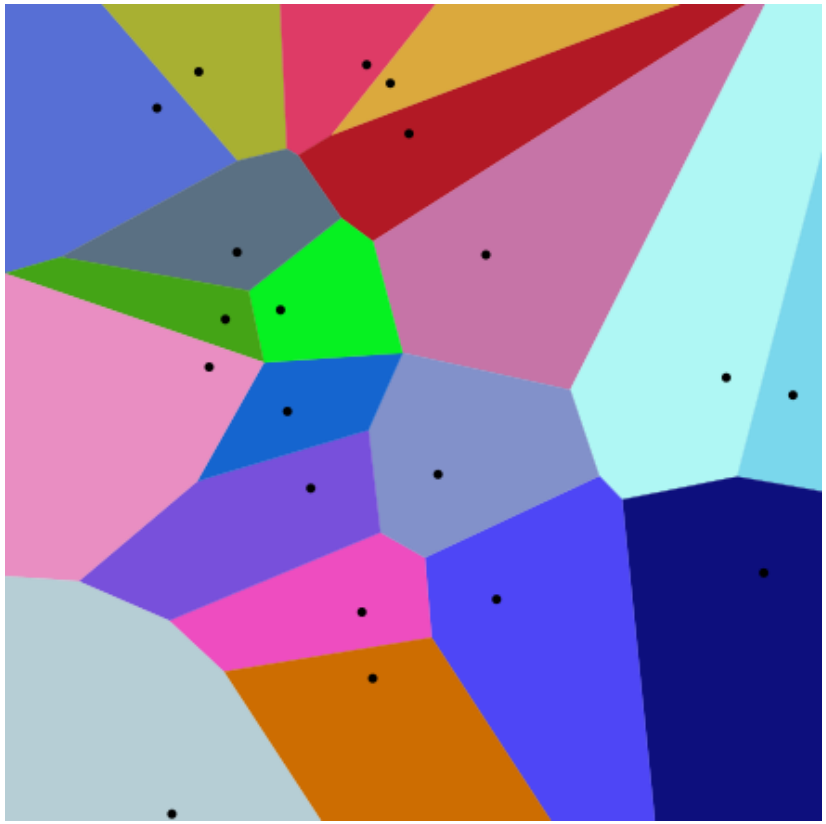
$L_p\text{-dist}((x_1,y_1),(x_1,y_1))=(|x_1-x_1|^p+|y_1-y_1|^p)^{1/p}$

$L_p\text{-dist}(v_1,v_2) = (|\Delta x|^p+|\Delta y|^p+|\Delta z|^p+|\Delta w|^p+\ldots)^{1/p}$

e.g., $L_1\text{-dist}((x_1,y_1),(x_1,y_1))=|x_1-x_1|+|y_1-y_1|$

Peter Dirichlet
1805-1859

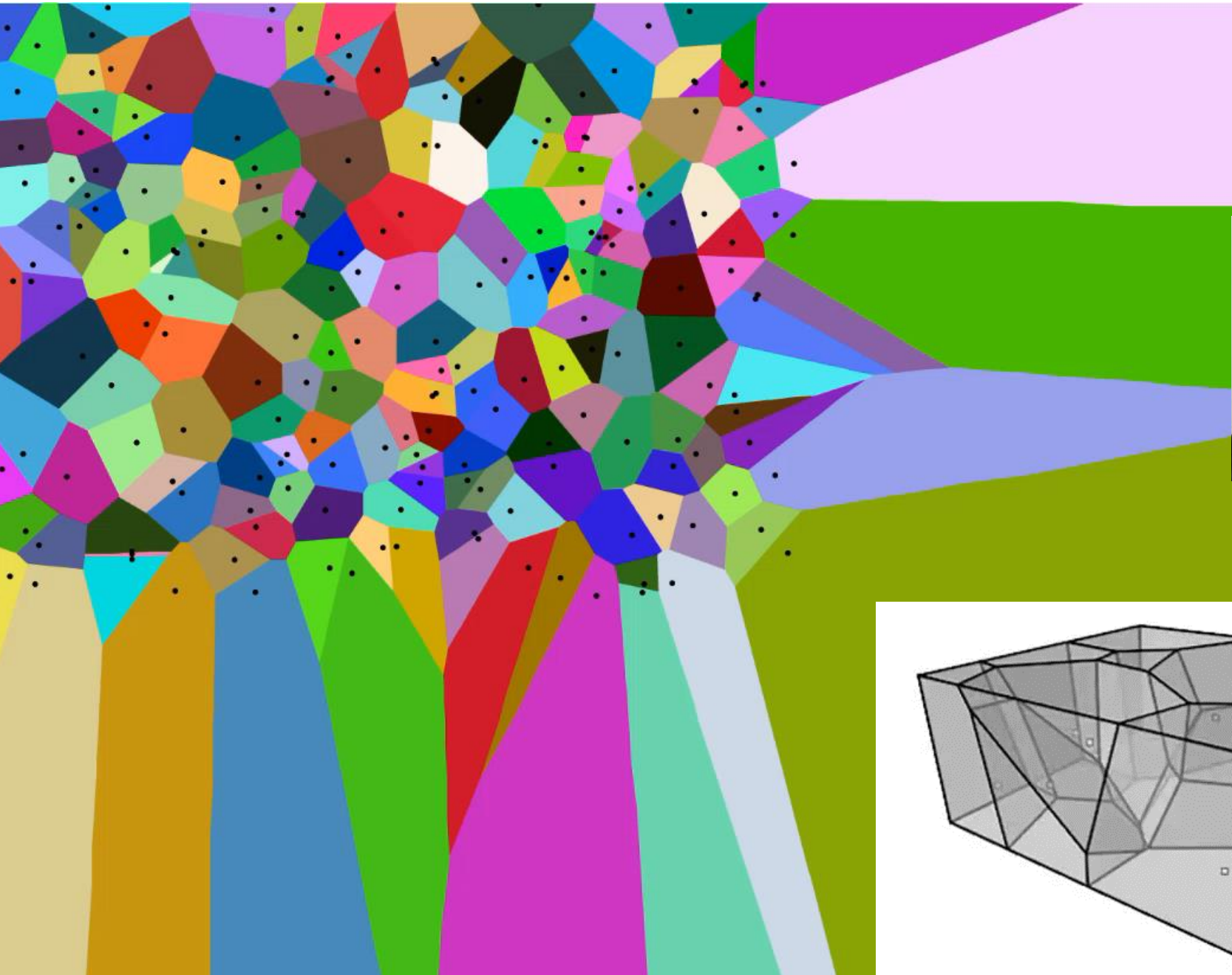
Georgy Voronoi
1868-1908



Euclidean metric (L_2)

Manhattan metric (L_1)

Dynamic Voronoi Diagrams



Peter Dirichlet
1805-1859

Georgy Voronoi
1868-1908

ISVD 2013

Saint Petersburg, Russia | July 8-10, 2013

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The 10th International Symposium on Voronoi Diagrams in Science and Engineering (ISVD 2013) will take place on July 8-10, 2013 at Saint Petersburg Academic University, Saint Petersburg, Russia. In frame of this Symposium, a specialized workshop "Voronoi Diagrams in Material Sciences and Molecular Biology" is planned, where applications of the method in natural sciences will be discussed.

Saint Petersburg Academic University

Saint Petersburg is the city where Georgy F. Voronoi studied, and both Georgy F. Voronoi and Boris N. Delaunay worked at the University nowadays called Saint Petersburg State University.

Georgy F. Voronoi
(1868-1908)

Saint Petersburg State University
(main building)

Boris N. Delaunay
(1890-1980)

In previous years, ISVD was held at Rutgers University, USA (2012), in Qingdao, China (2011), Québec, Canada (2010), Copenhagen, Denmark (2009), Kyiv, Ukraine (2008), Pontypridd, Wales, United Kingdom (2007), Banff, Canada (2006), Seoul, South Korea (2005), and Tokyo, Japan (2004).

International Symposium on Voronoi Diagrams in Science and Engineering

DIMACS, Rutgers University
Piscataway, New Jersey, USA **June 27 – 29, 2012**
www.isvd12.rutgers.edu

DIMACS
Center for Discrete Mathematics & Theoretical Computer Science
Founded as a National Science Foundation Science and Technology Center

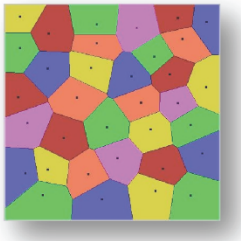
NSF

Conference Announcement:

The 9th International Symposium on Voronoi Diagrams in Science and Engineering (ISVD 2012) will take place on **June 27 – 29, 2012** at the Computing Research and Education (CoRE) building on the Busch Campus of Rutgers University in New Jersey, USA. The conference provides a forum for practitioners devoted to research on Voronoi diagrams and its applications to real-world problems that originate in a variety of scientific and engineering disciplines. **Early Registration Deadline: May 7, 2012**.

Call for Papers:

The sponsors seek submission of papers on original and unpublished research (up to 10 pages) in areas of Voronoi diagrams and their applications related to the following topics: Mathematical aspects of Voronoi diagrams and Delaunay graphs; Generalizations of Voronoi diagrams and Delaunay graphs; Algorithms and implementation of Voronoi diagrams and Delaunay graphs; Conceptual and logical data models based on Voronoi diagrams; Terrain modeling and meshes using Delaunay graphs; Visualization, animation, and morphing; Pattern analysis and recognition; Motion planning, navigation, collision detection and obstacle avoidance ; Network analysis and communication; Clustering using nearest-neighbor approach and weighted distance function; Computer modeling and simulation; Spatial and temporal statistics; Image processing; Biological, molecular, and physical modeling; Voronoi diagrams in astronomy, bioinformatics, geography, chemistry, material science, renewable energy, location science, solid modeling, and operations research; original research in polynomiography and biometrics related to Voronoi diagrams.



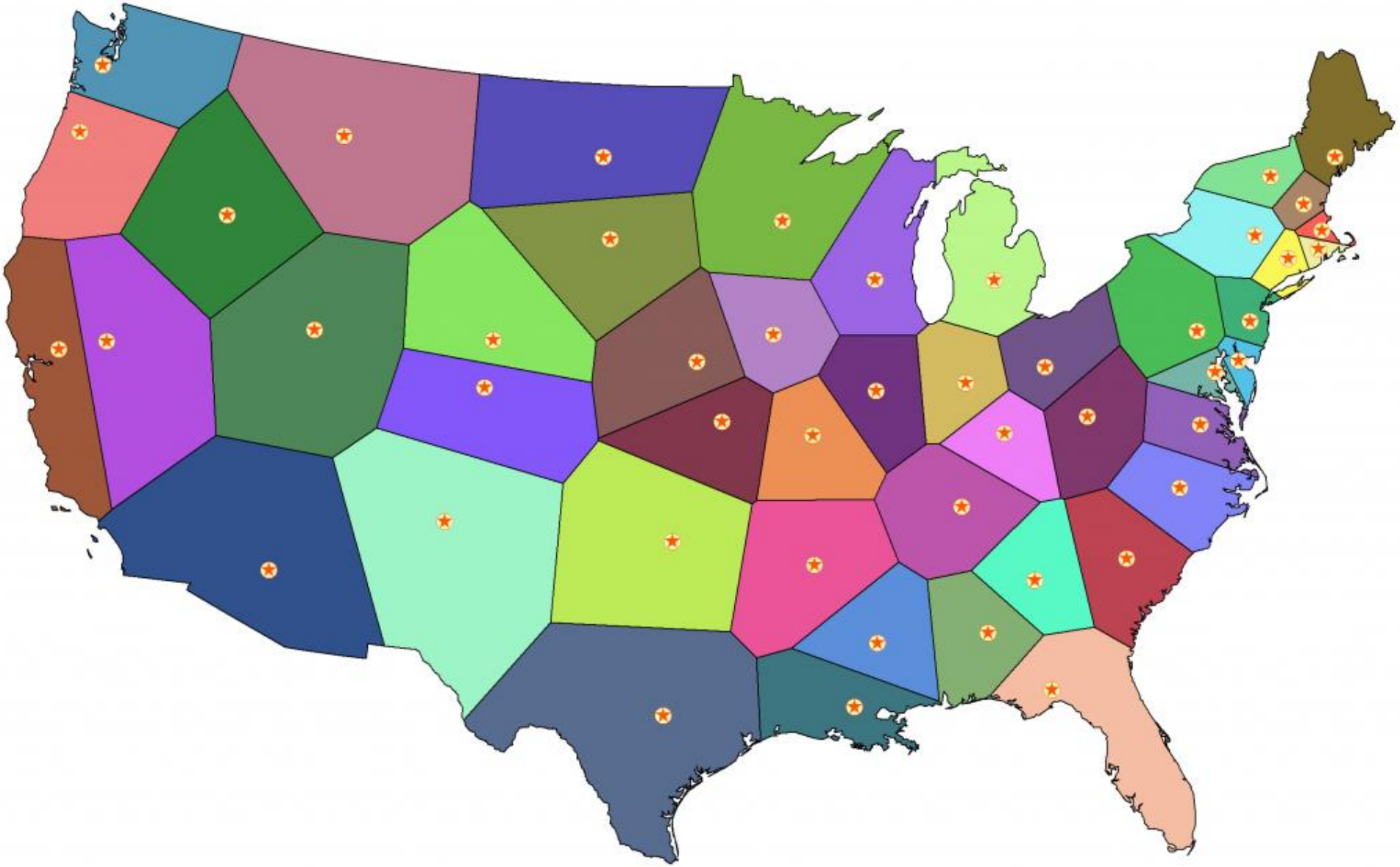
Authors are also invited to submit Voronoi art posters (up to A0 format) or videos exhibiting nice applications of Voronoi diagrams to be presented on June 26th as a special Voronoi art exhibition.

All submitted papers will be refereed for quality, originality, and relevance by the International Programme Committee. Accepted papers will be published by the Conference Publishing Services in the proceedings of the symposium (with ISBN) and will also be available online. Selected papers will be published in a special issue of the Springer Journal of Transactions on Computational Science and other journals.

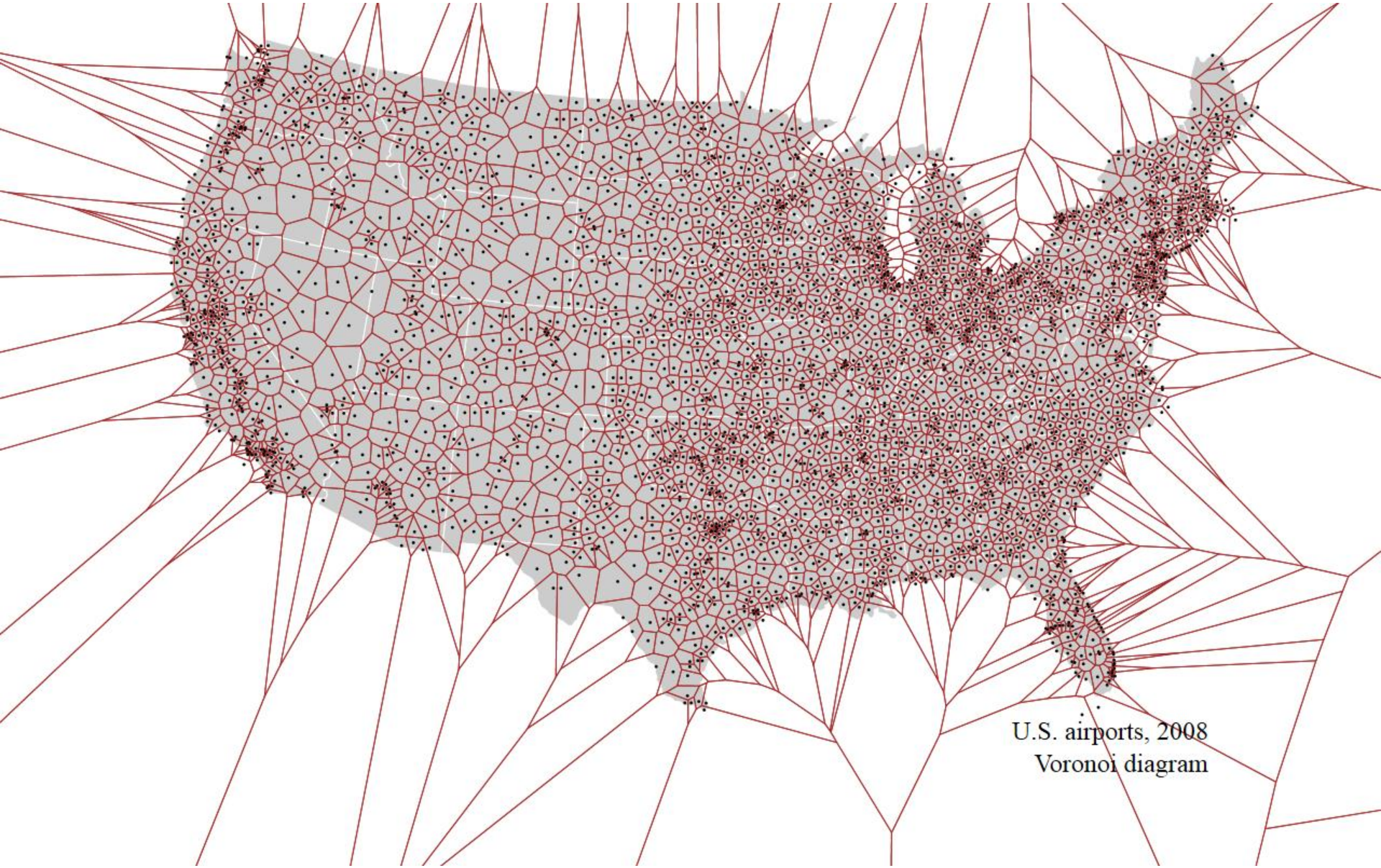
Full Paper Submission Deadline: April 07,2012

The Conference is organized by the Center for Discrete Mathematics and Theoretical Computer Science (DIMACS) and the Rutgers

Voronoi Diagram of State Capitals



Voronoi Diagram of U.S. Airports



Voronoi Furniture

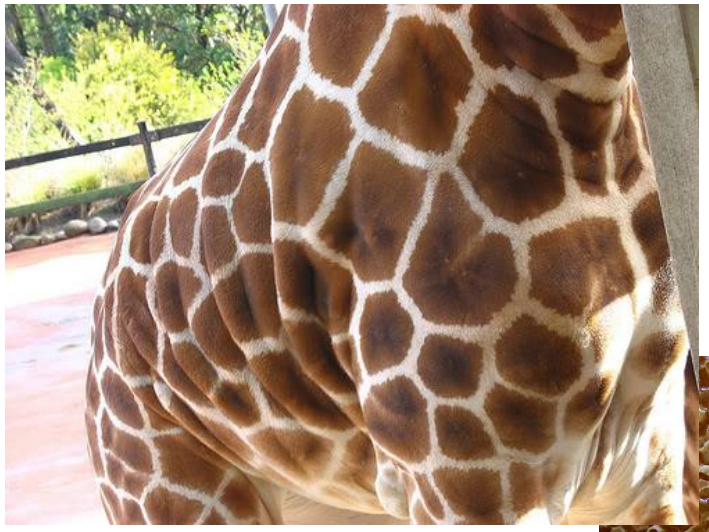
Voronoi Furniture

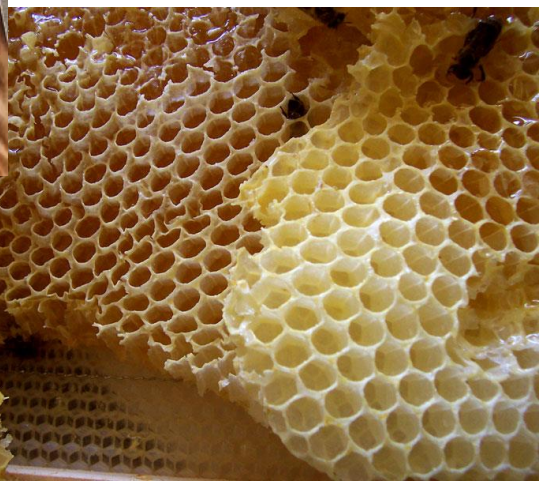
Voronoi Architecture

Voronoi Architecture

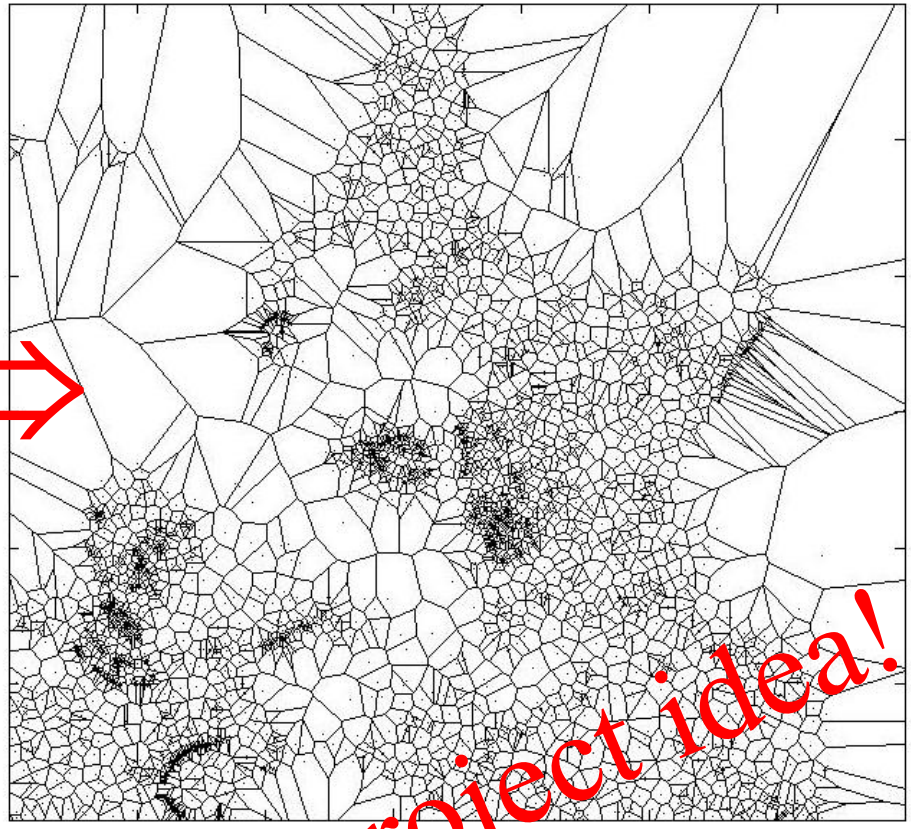
Voronoi Architecture

Voronoi Diagrams in Nature



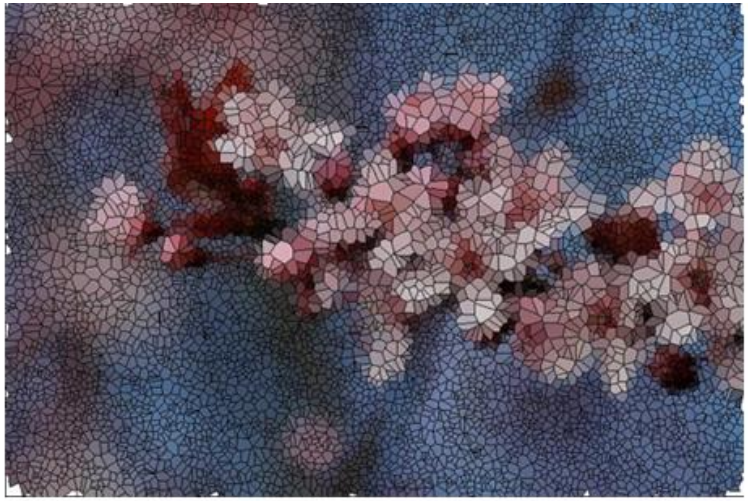


Voronoi Art



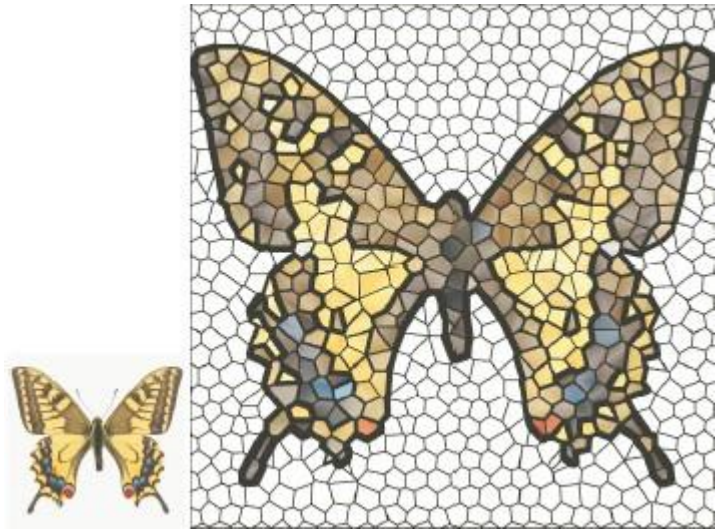
Term project idea!

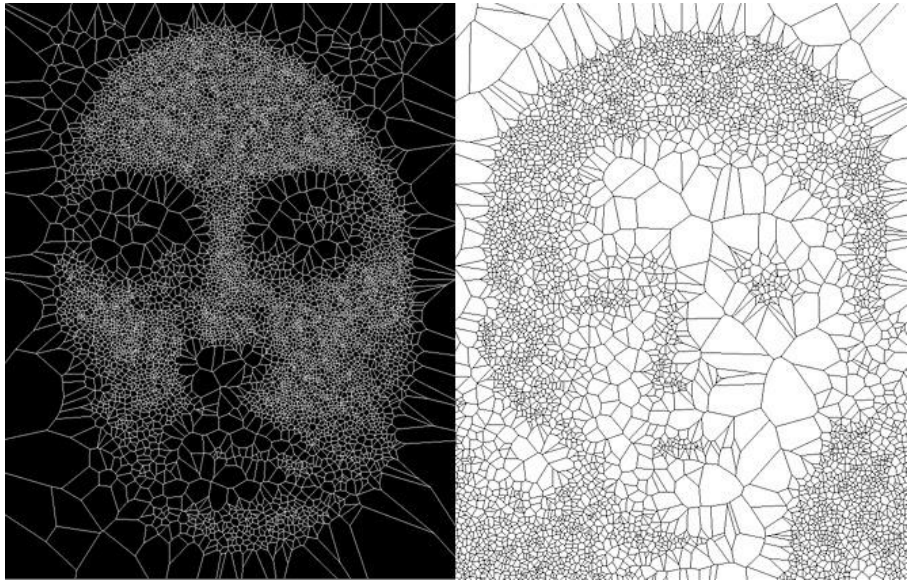
Voronoi Art

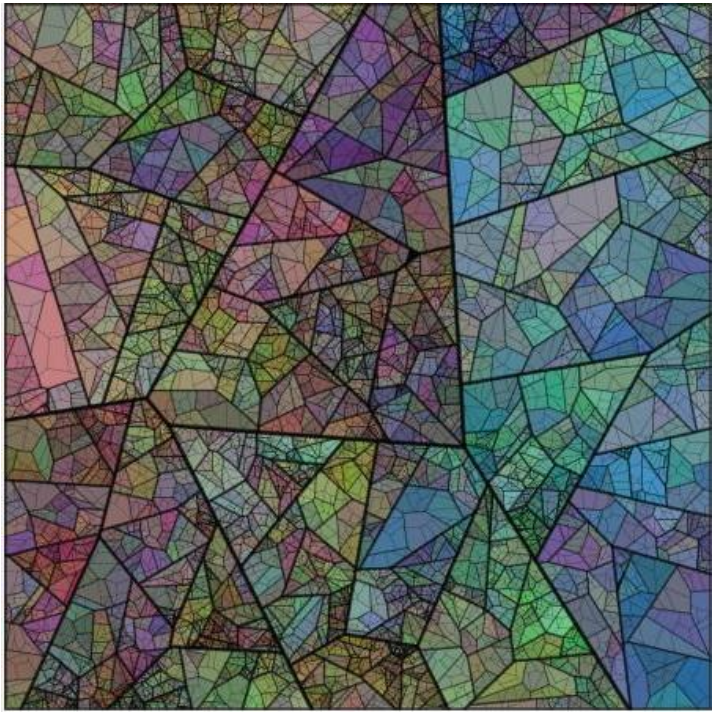


Term project idea!

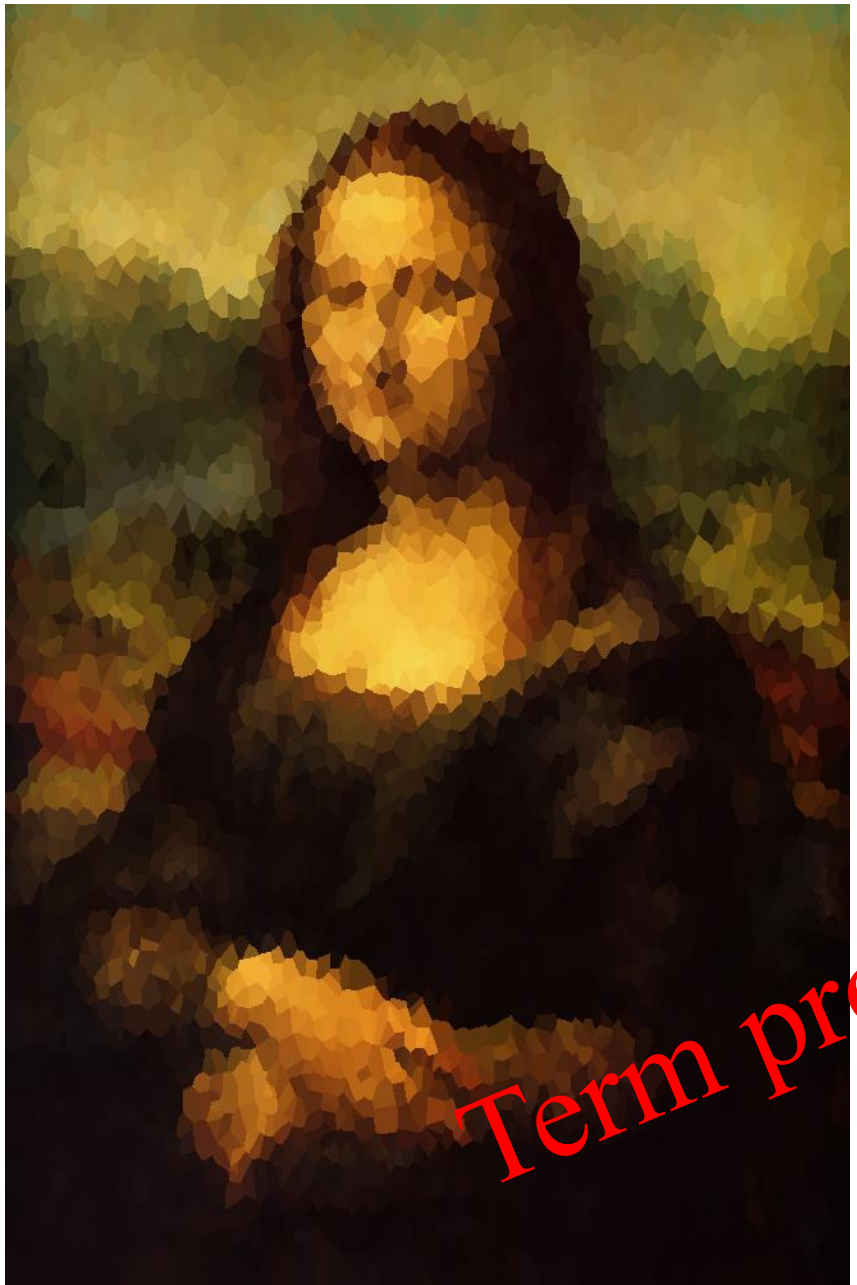
Voronoi Art





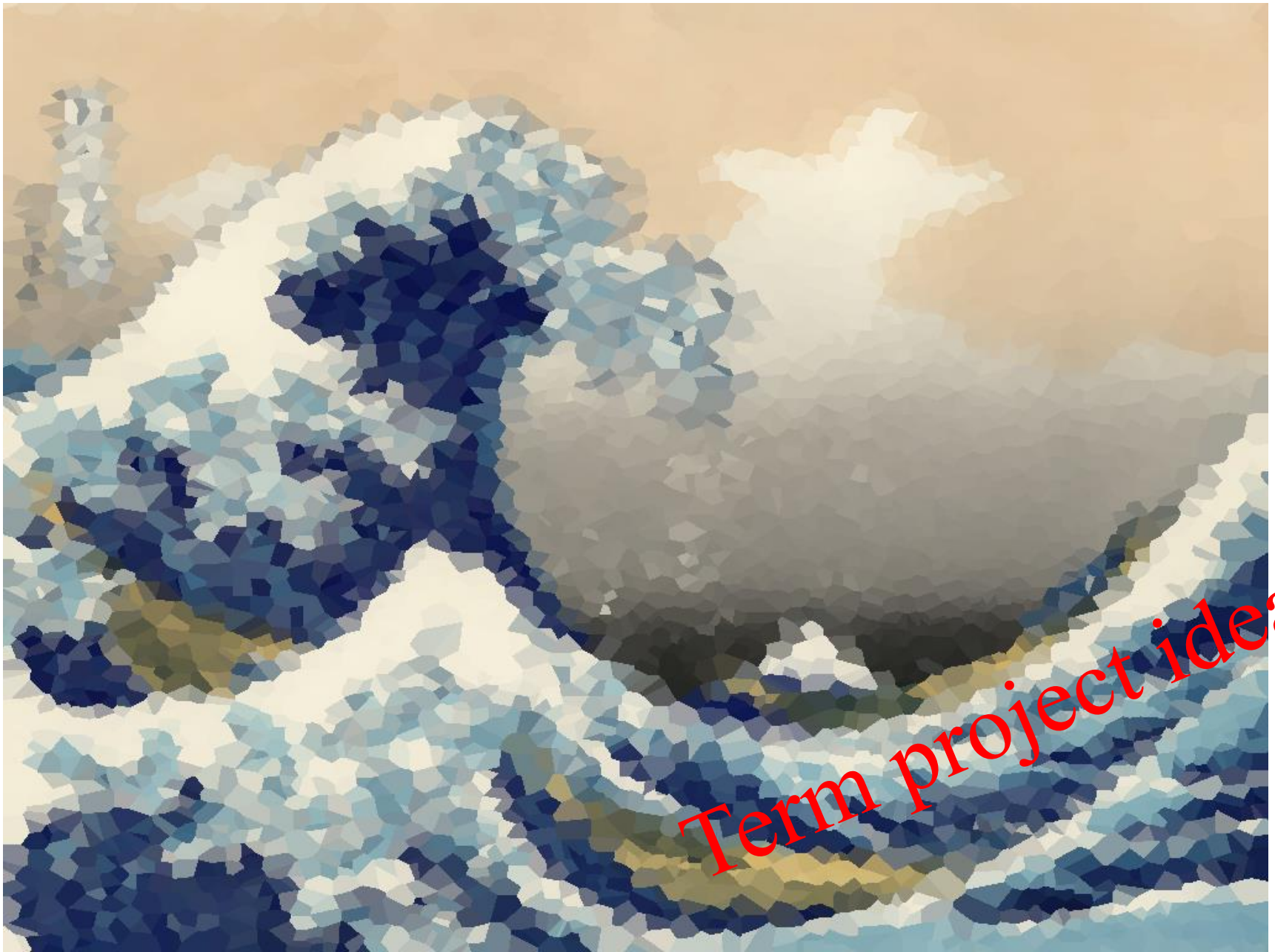


Voronoi Art

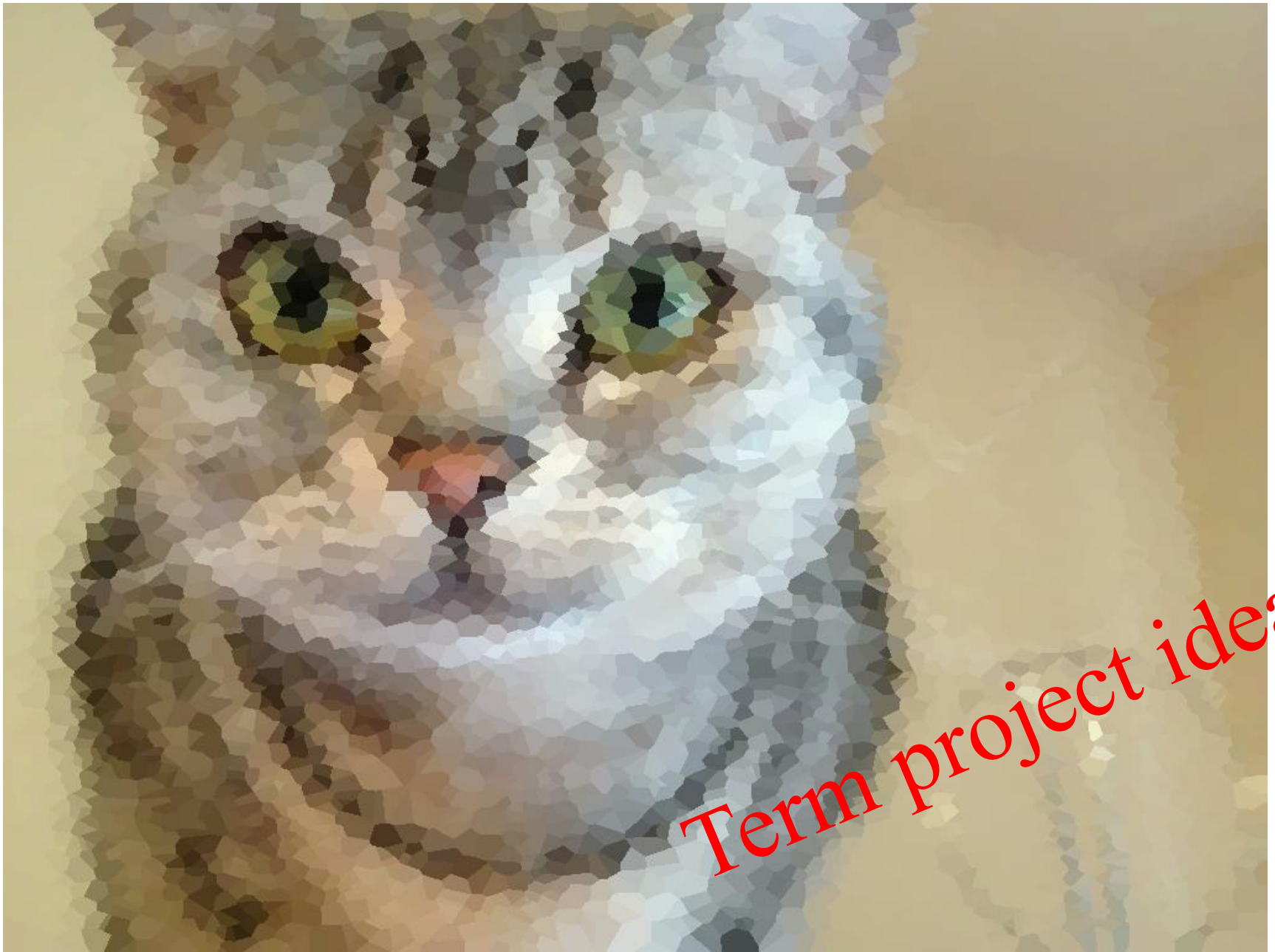


Term project idea!

Voronoi Art



Voronoi Art



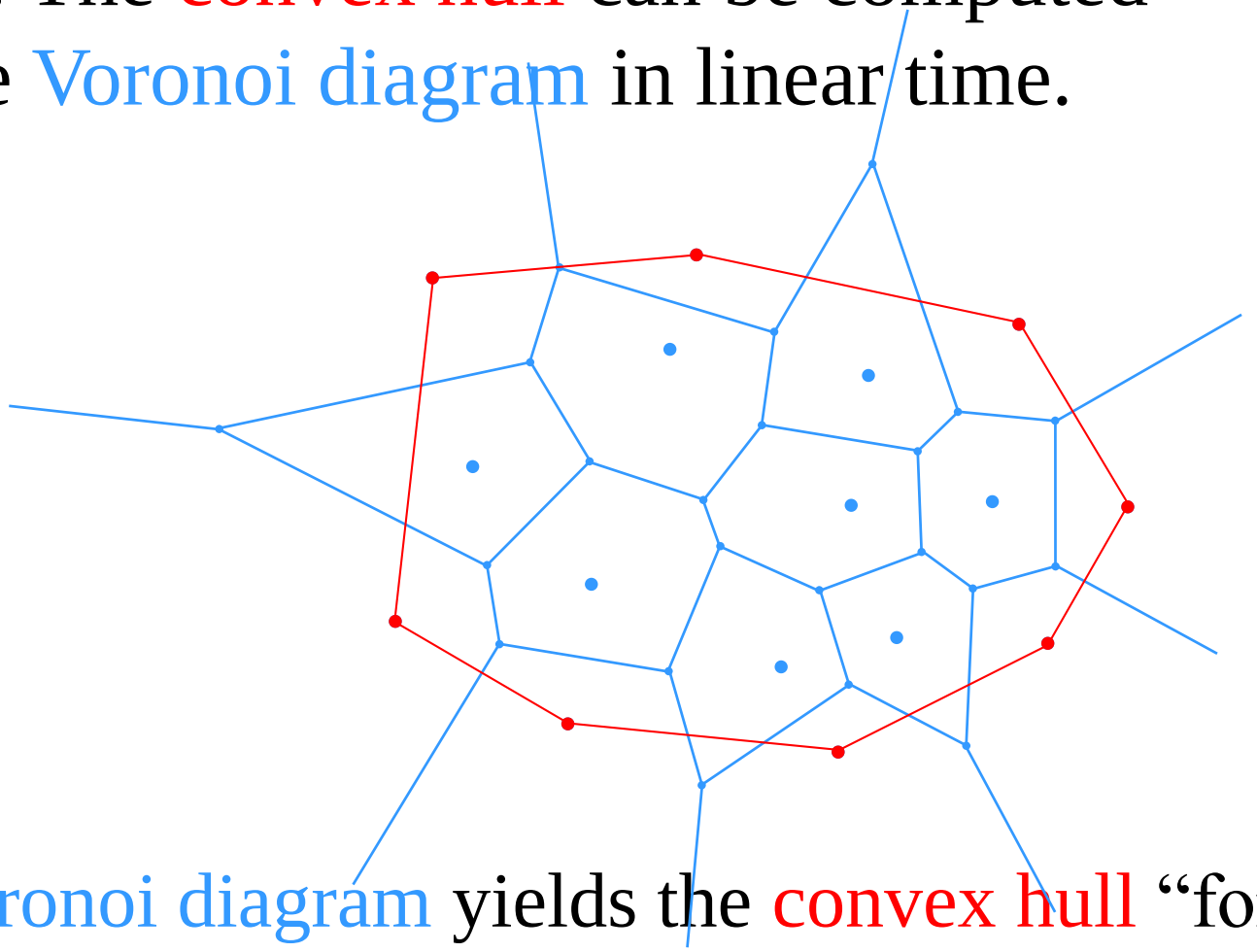
Voronoi Diagram Properties

Peter Dirichlet
1805-1859

Georgy Voronoi
1868-1908

Theorem: A Voronoi cell is unbounded if and only if it's point is on the convex hull.

Corollary: The convex hull can be computed from the Voronoi diagram in linear time.



$\Rightarrow$ The Voronoi diagram yields the convex hull "for free"

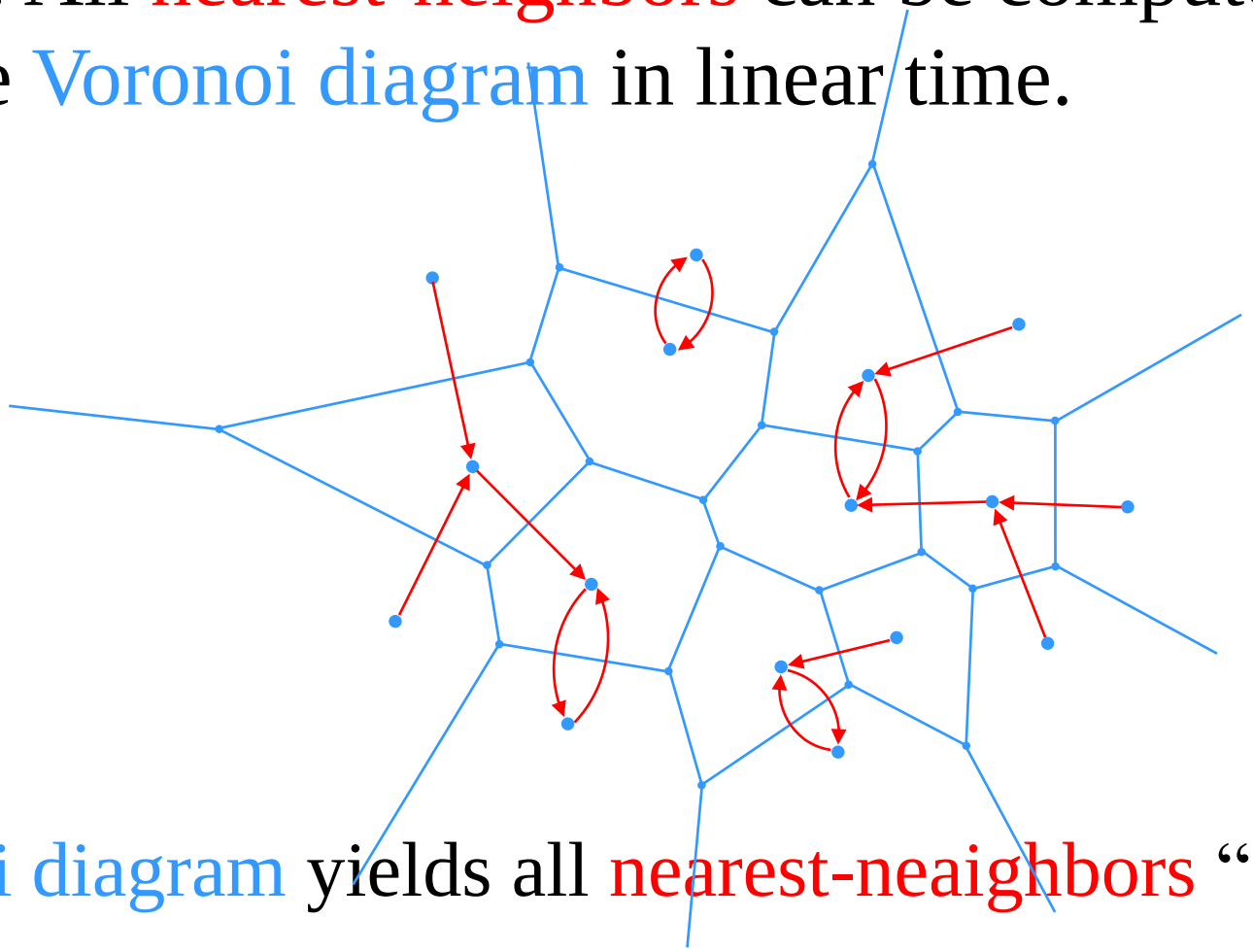
Voronoi Diagram Properties

Peter Dirichlet
1805-1859

Theorem: Every nearest neighbor of a point defines an edge of the Voronoi diagram.

Corollary: All nearest-neighbors can be computed from the Voronoi diagram in linear time.

Georgy Voronoi
1868-1908



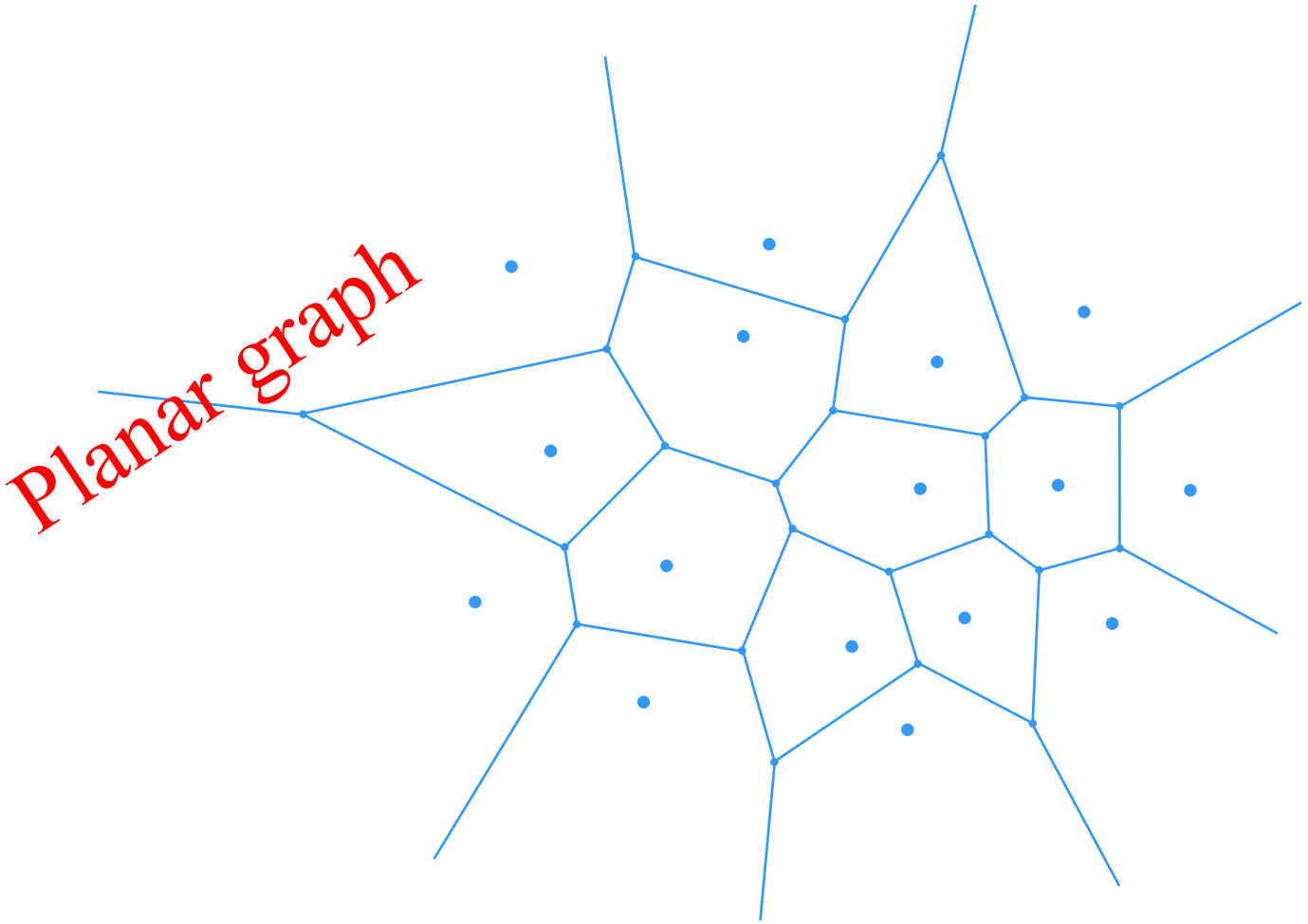
$\Rightarrow$ Voronoi diagram yields all nearest-neaighbors “for free”

Voronoi Diagram Properties

Theorem: A Voronoi diagram on n points has at most 2n-5 vertices and 3n-6 edges.

Peter Dirichlet
1805-1859

Georgy Voronoi
1868-1908



Voronoi Diagram Properties

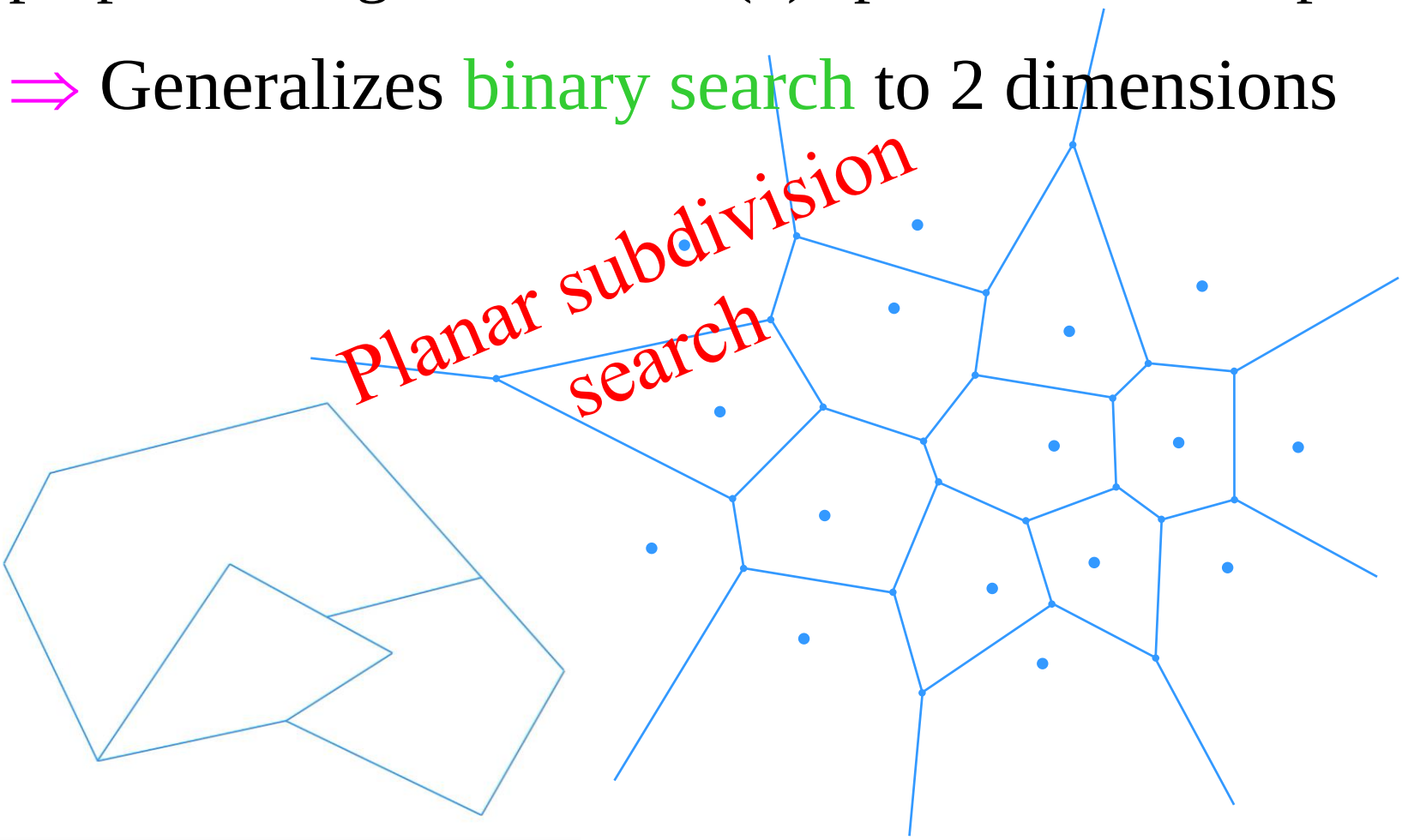
Peter Dirichlet
1805-1859

Theorem: The Voronoi diagram enables nearest neighbor search in O(log n) time, using O(n log n) preprocessing time, and O(n) space, which is optimal.

Georgy Voronoi
1868-1908

⇒ Generalizes binary search to 2 dimensions

Planar subdivision search



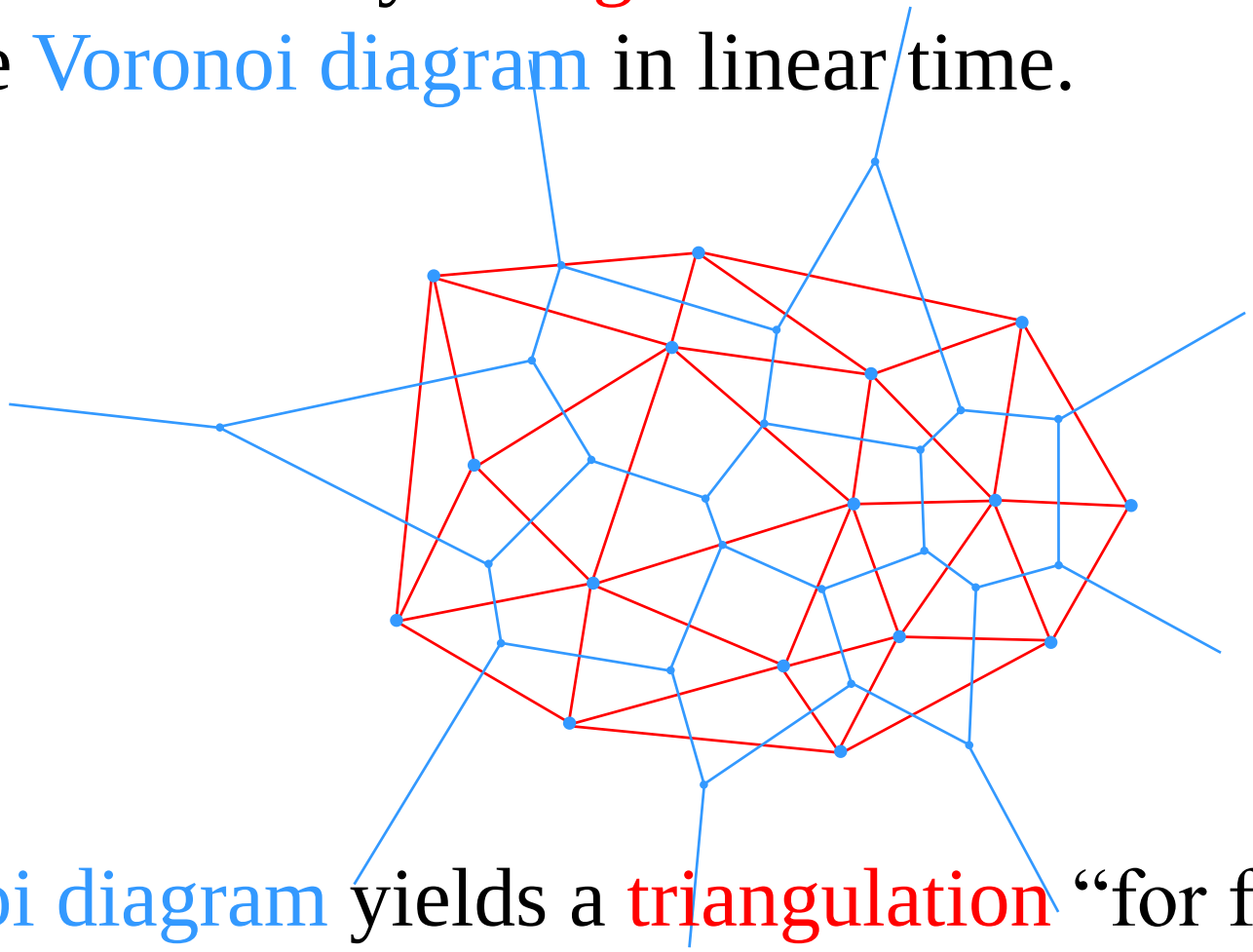
Voronoi Diagram Properties

Theorem: Connecting points of neighboring Voronoi diagram cells forms a triangulation.

Peter Dirichlet
1805-1859

Corollary: A Delanuay triangulation can be computed from the Voronoi diagram in linear time.

Georgy Voronoi
1868-1908



Boris Delaunay
1890-1980

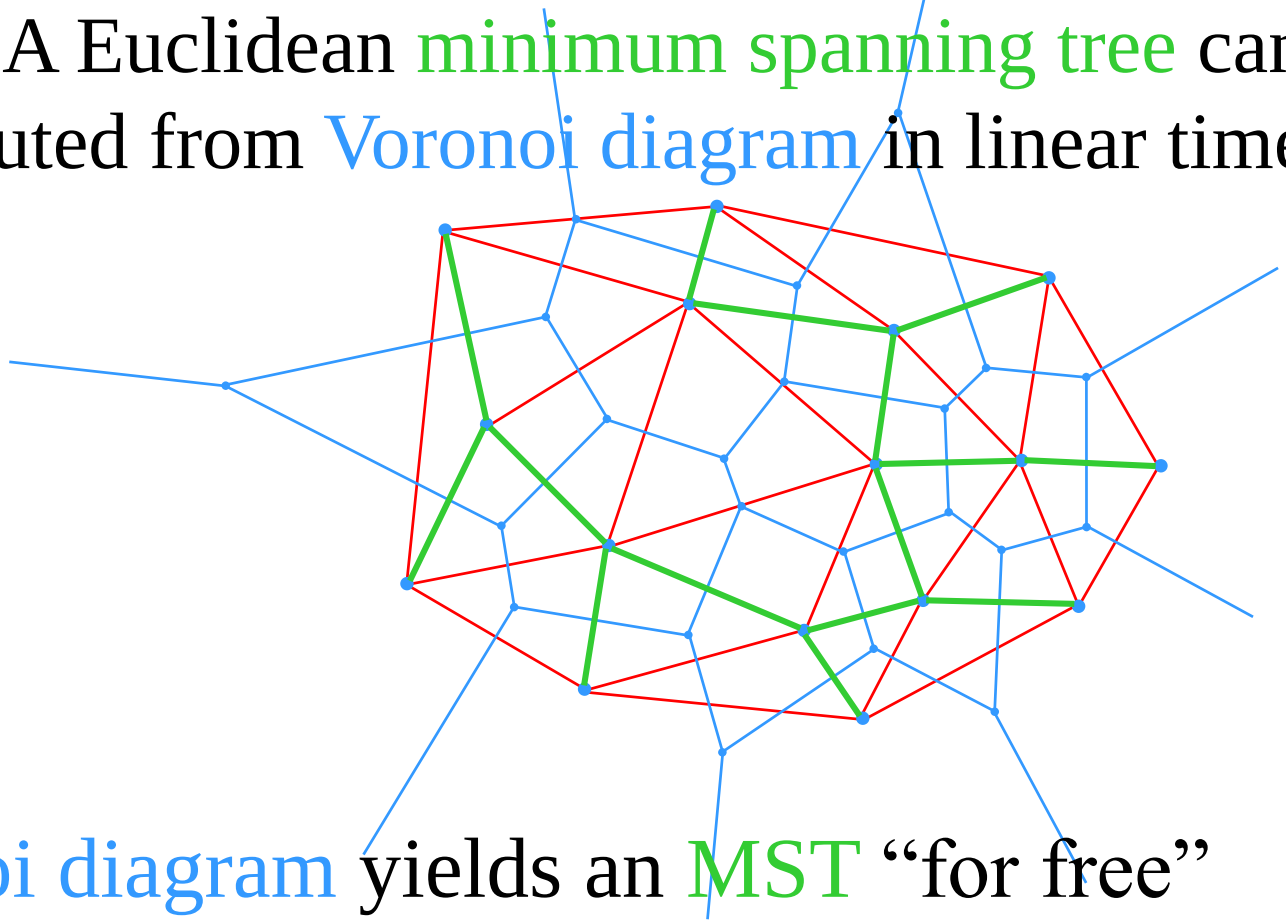
⇒ Voronoi diagram yields a triangulation “for free”

Voronoi Diagram Properties

Theorem: A Delanuay triangulation maximizes the minimum angle over all triangulations.

Theorem: A Euclidean minimum spanning tree is a subset of the Delanuay triangulation.

Corollary: A Euclidean minimum spanning tree can be computed from Voronoi diagram in linear time.



$\Rightarrow$ Voronoi diagram yields an MST "for free"

Peter Dirichlet
1805-1859

Georgy Voronoi
1868-1908

Boris Delaunay
1890-1980

Voronoi Diagram Properties

Thm: Convex hull doable in Θ(n log n) time.
Thm: Nearest neighbors doable in Θ(n log n) time.
Thm: Closest pair doable in Θ(n log n) time.
Thm: Triangulation doable in Θ(n log n) time.
Thm: Euclidean MST doable in Θ(n log n) time.

Thm: Convex hull requires Ω(n log n) time.
Thm: Nearest neighbors require Ω(n log n) time.
Thm: Closest pair requires Ω(n log n time).
Thm: Triangulation requires Ω(n log n time).
Thm: Euclidean MST requires Ω(n log n) time).

Thm: Voronoi diagram can be used to solve all of the above problems in O(n log n) time (and linear time given the Voronoi diagram).


Peter Dirichlet
1805-1859


Georgy Voronoi
1868-1908

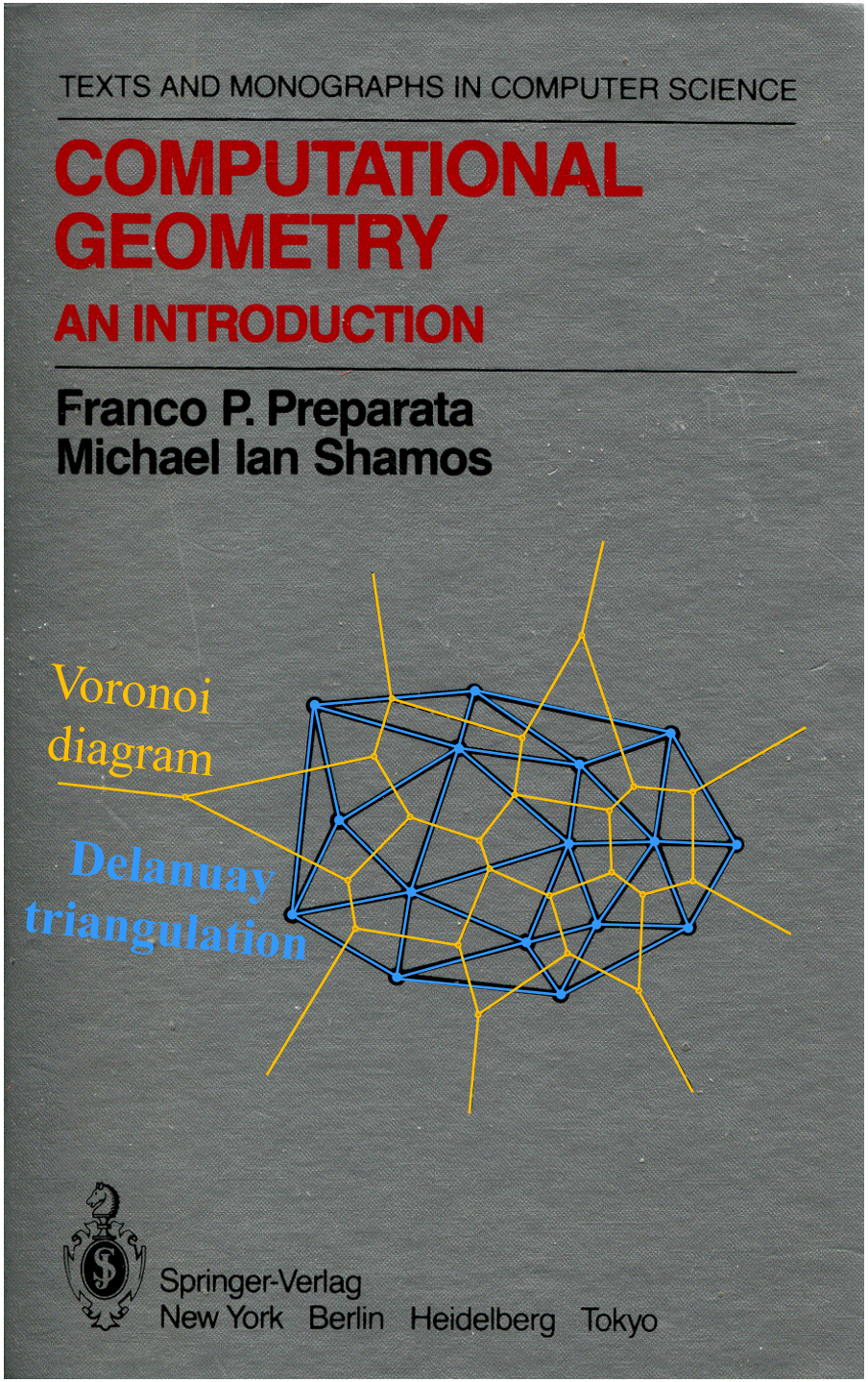

Boris Delaunay
1890-1980


Michael Shamos
1947-

Peter Dirichlet
1805-1859

Georgy Voronoi
1868-1908

Boris Delaunay
1890-1980



Michael Shamos

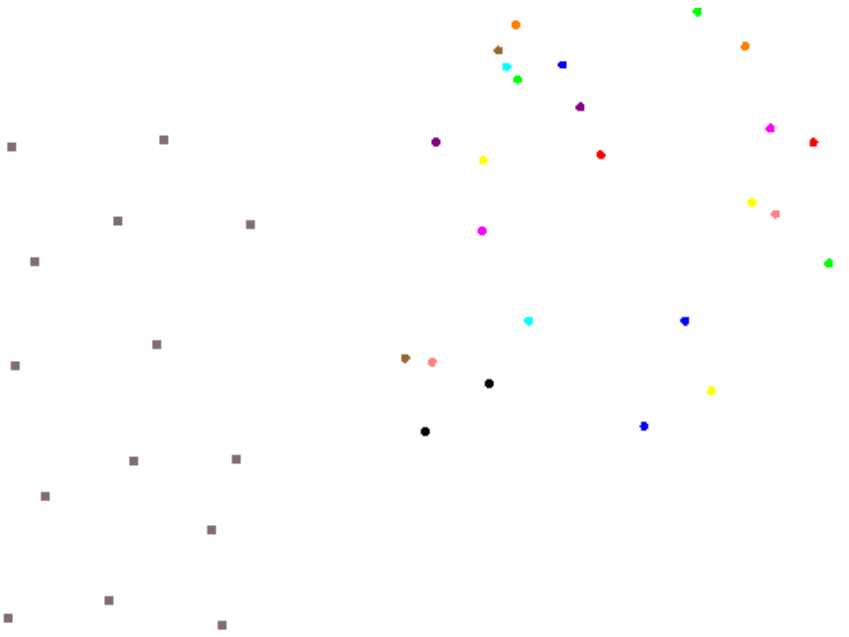
Franco Preparata

Voronoi Diagram Algorithms

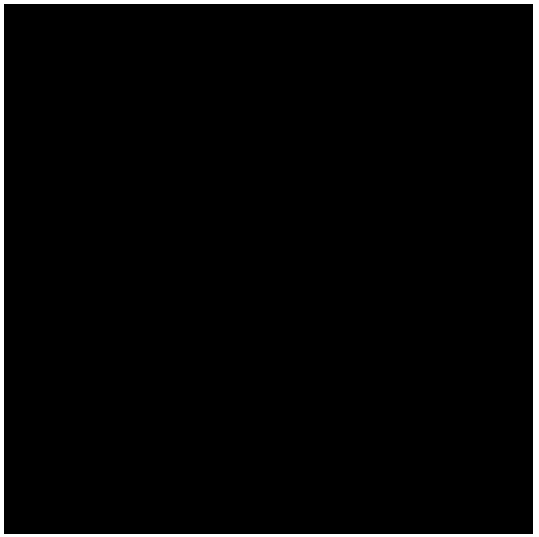
Input: set of n points

Output: Voronoi diagram

- Discrete case / bitmap:



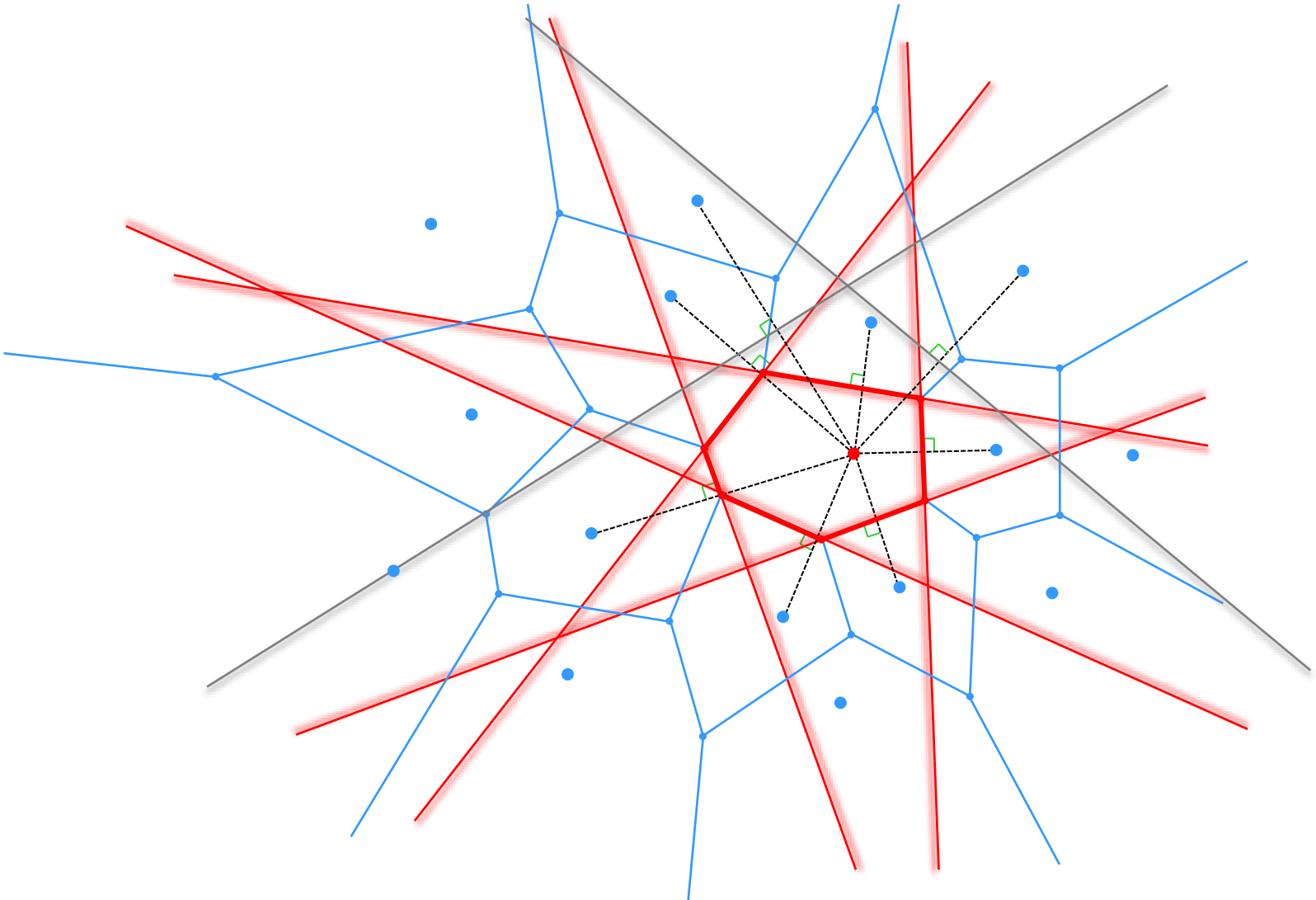
Expand breadth-first waves
from all points in parallel
Mark expanded pixels
uniquely for each wave
When waves collide,
freeze all collision points



- Total time is O(bitmap size)
- Time is independent of #points n
- Identifies pixels, not polygons

Voronoi Diagrams

Theorem: Voronoi cell of a point is ∩ of all half-planes induced by the perpendicular bisectors w.r.t. all other points.



Voronoi Diagram Algorithms

Input: set of n points

Output: Voronoi diagram

- Continuous case / polygons:

> Compute Voronoi cell by
> intersecting n half-planes

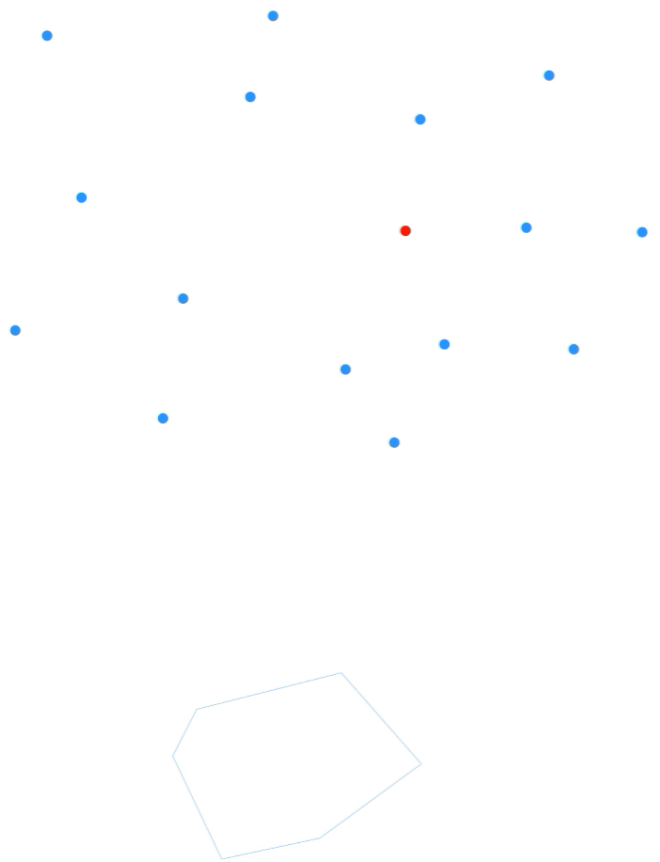
- Time per cell is $O(n^2)$

> For each point
> Compute Voronoi cell
> Voronoi diagram is their union

- Total time is $O(n^3)$

Theorem: ∩ of n half-planes is computable in $O(n \log n)$ time.

- Total Voronoi diagram time improves to $O(n^2 \log n)$



Voronoi Diagram Algorithms

Theorem [Shamos]: Voronoi diagrams in the plane can be computed in $\Theta(n \log n)$ time.


Michael Shamos

Idea: divide-and-conquer (like MergeSort)

- Complex merge step

Theorem [Fortune]: Voronoi diagrams in the plane can be computed in $\Theta(n \log n)$ time.

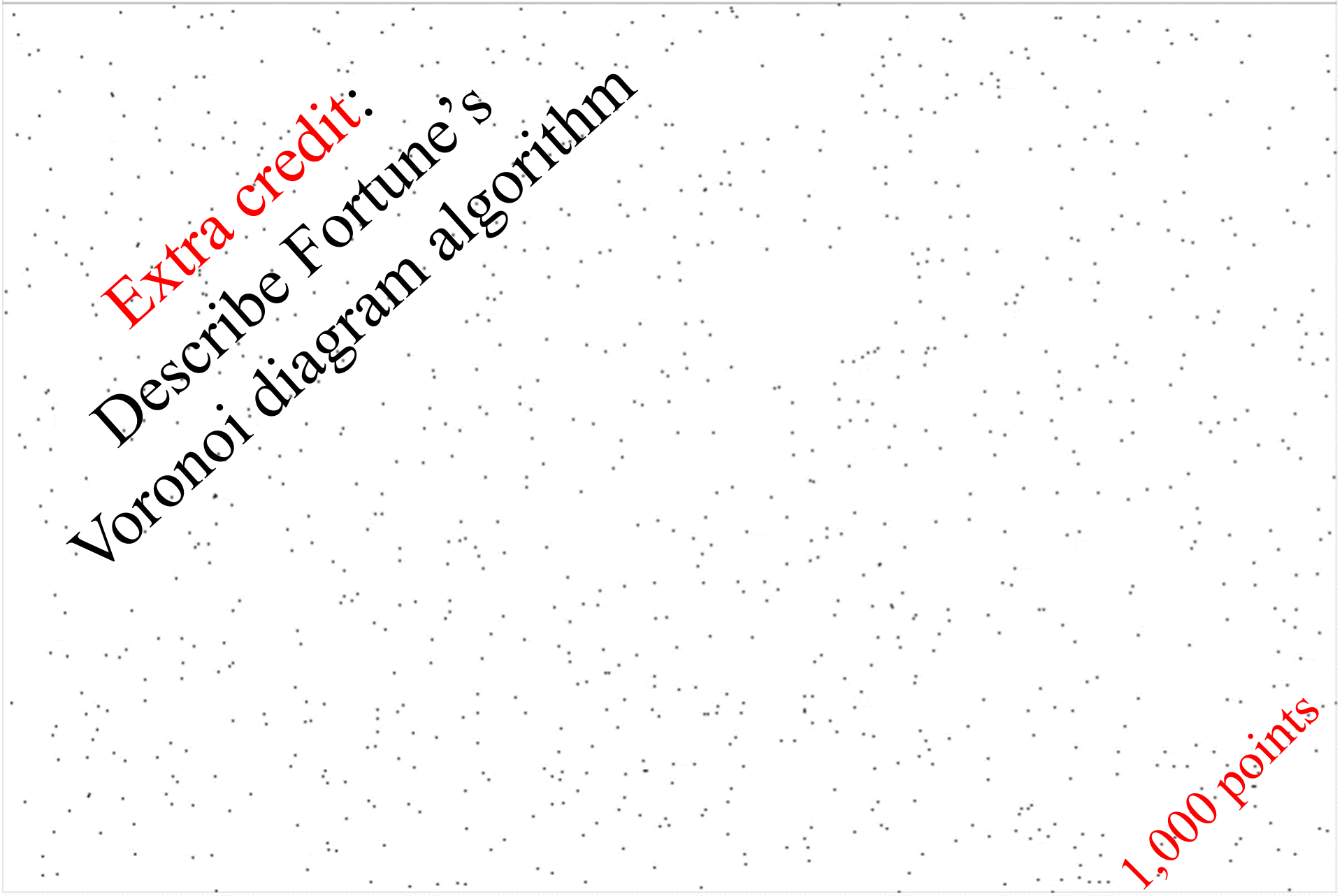

Steven Fortune

Idea: sweep line using parabolas

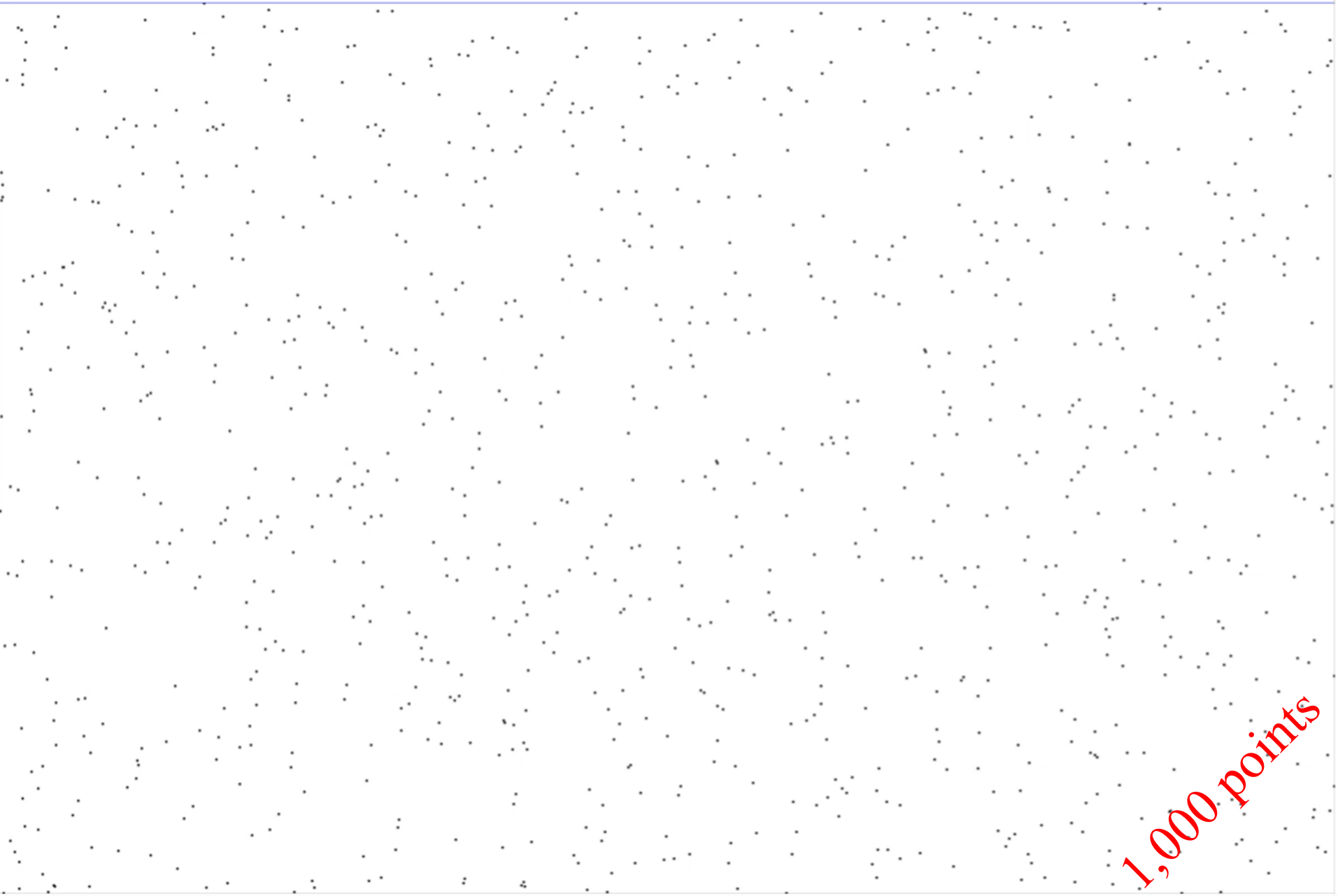
- [Fortune] is simpler than [Shamos]
- [Fortune] Generalizes to e.g, disks

Theorem: Voronoi diagrams require $\Omega(n \log n)$ time to compute.

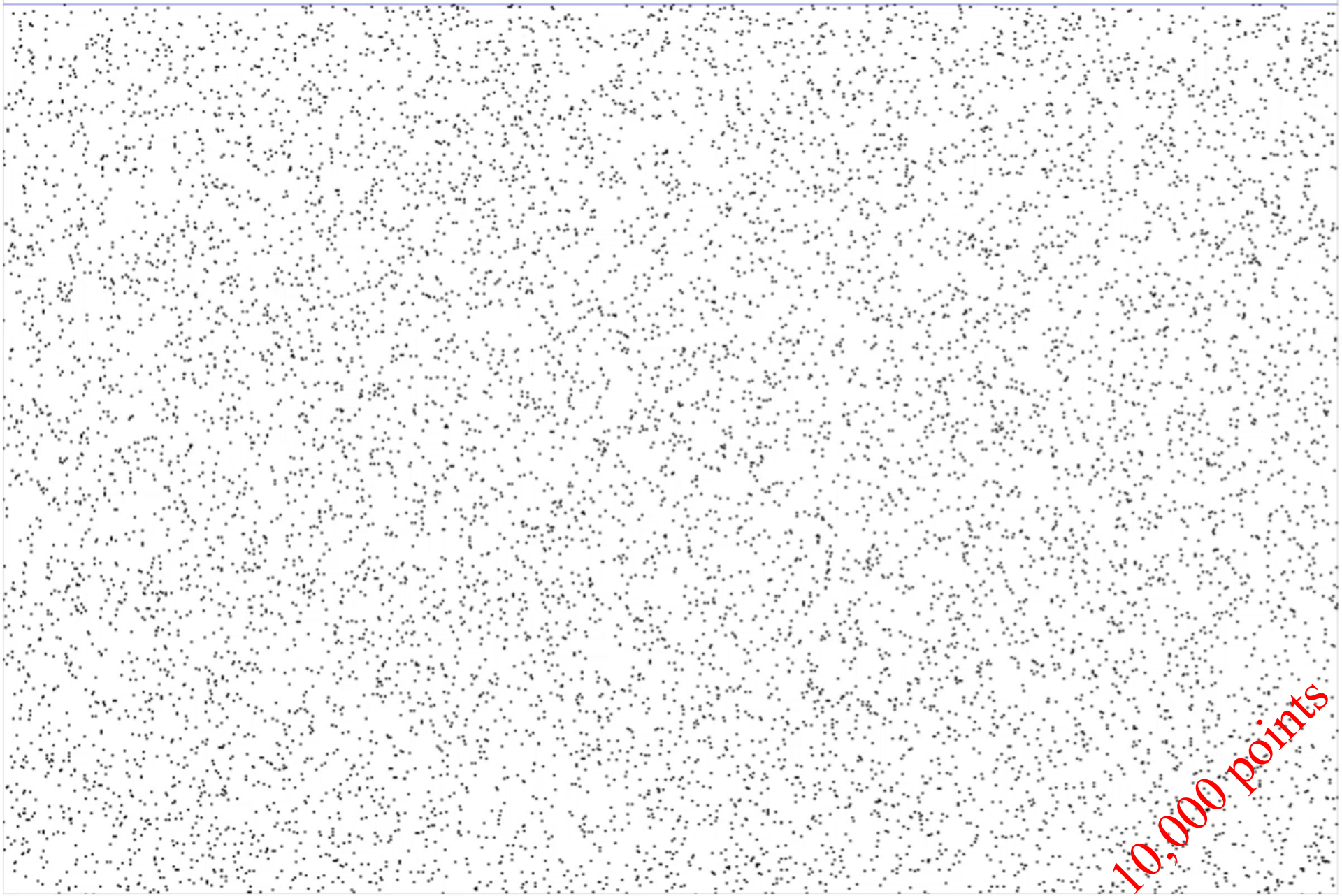
Voronoi Diagram Algorithm [Fortune]



Voronoi Diagram Algorithm [Fortune]



Voronoi Diagram Algorithm [Fortune]



Problem: Solve the following equation for X:

$$X^{X^{X^{X^{\cdots}}}} = 2$$

where the stack of exponentiated x's extends forever.

- What approaches fail?
- What techniques work and why?
- Lessons and generalizations

"Mr. Osborne, may I be excused? My brain is full."

Problem: Prove that there are an infinity of primes.

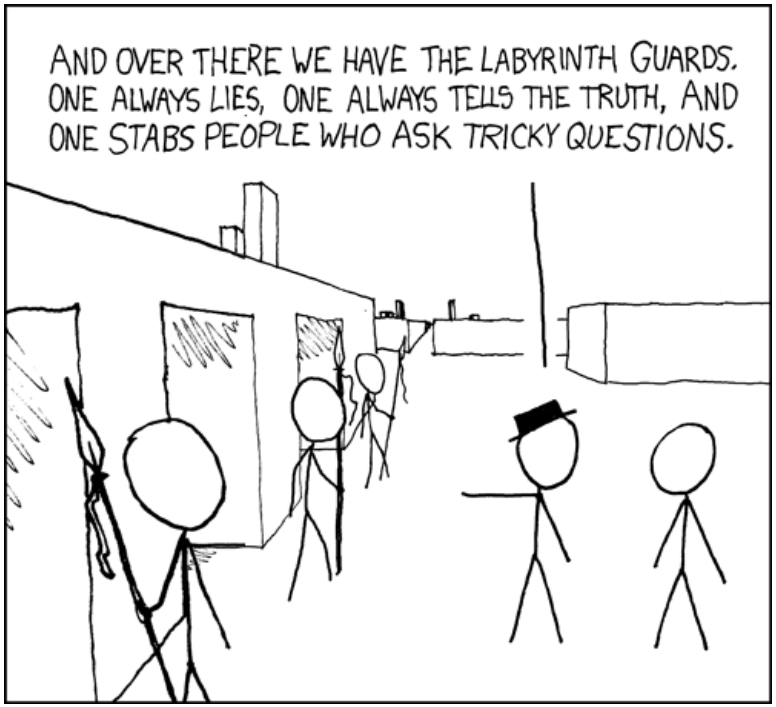
Extra Credit: Find a short, induction-free proof.

- What approaches fail?
- What techniques work and why?
- Lessons and generalizations

Problem: True or false: there are arbitrary long blocks of consecutive composite integers (i.e., big "prime deserts")

Extra Credit: find a short, induction-free proof.

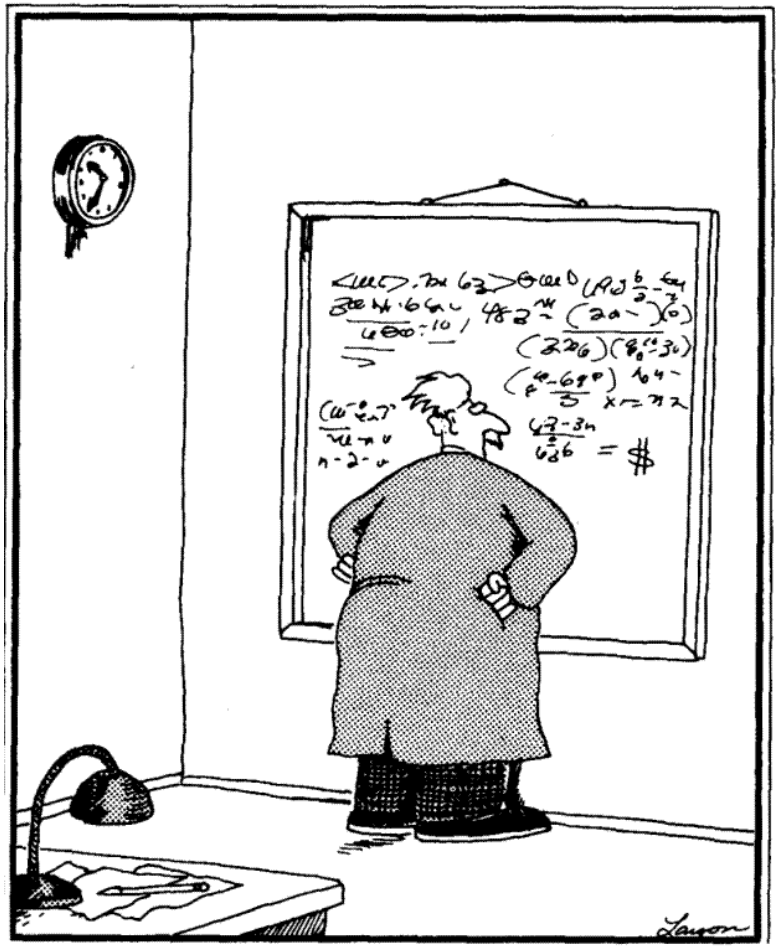
- What approaches fail?
- What techniques work and why?
- Lessons and generalizations



Problem: Prove that $\sqrt{2}$ is irrational.

Extra Credit: find a short, induction-free proof.

- What approaches fail?
- What techniques work and why?
- Lessons and generalizations

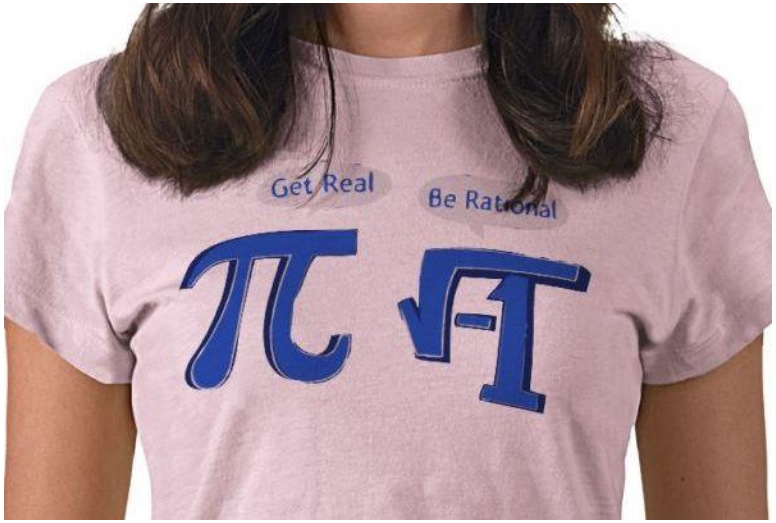


Einstein discovers that time is actually money.

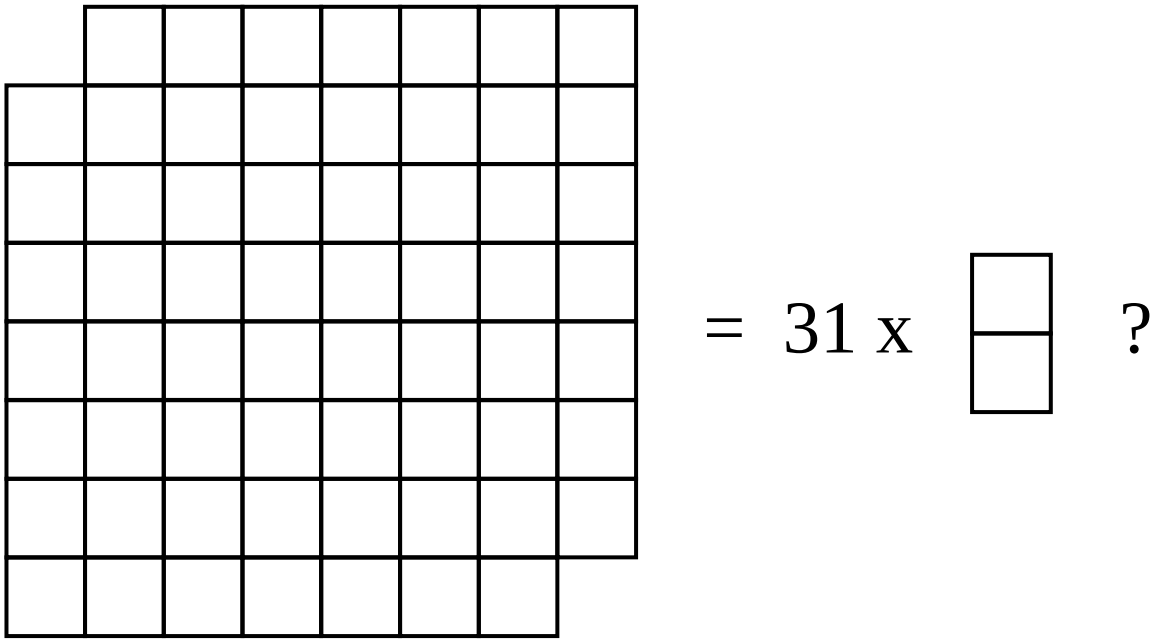
Problem: Does exponentiation preserve irrationality?
i.e., are there two irrational numbers x and y such that x^y is rational?

Extra Credit: find a short, induction-free proof.

- What approaches fail?
- What techniques work and why?
- Lessons and generalizations



Problem: Can an 8x8 board with two opposite corners missing be tiles with 31 dominoes?



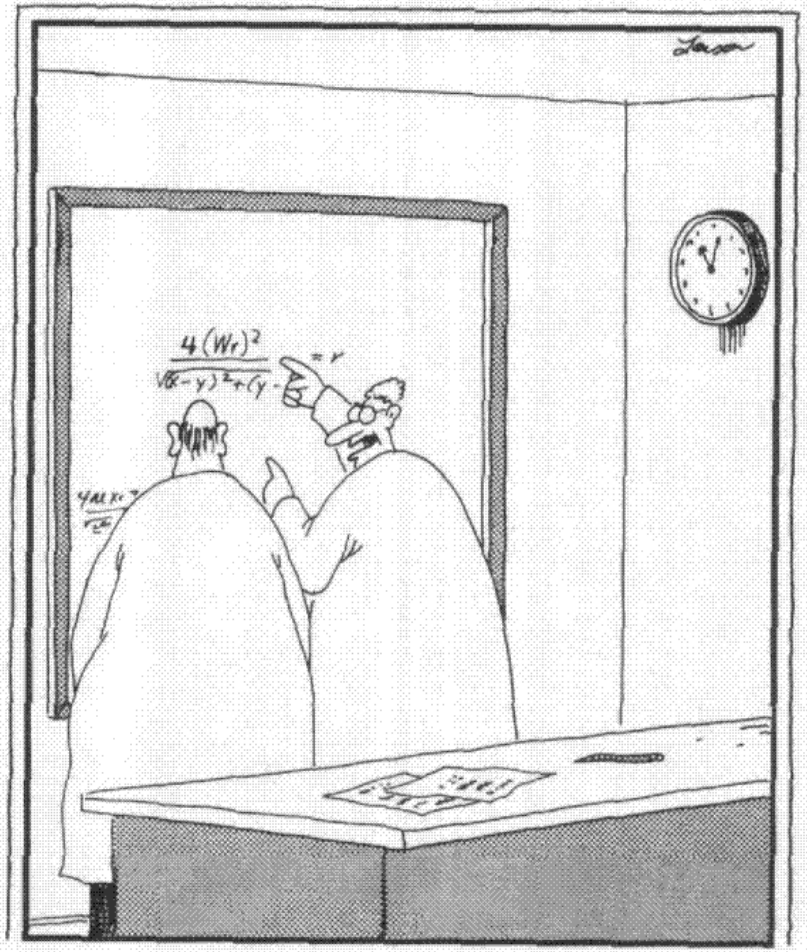
- What approaches fail?
- What techniques work and why?
- Lessons and generalizations

Problem: $1^3 + 2^3 + 3^3 + 4^3 + \ldots + n^3 = ?$

$$\sum_{i=1}^{n} i^3 = ?$$

Extra Credit:

find a short, geometric, induction-free proof.



"Yes, yes, I *know* that, Sidney … *everybody* knows *that*! … But look: *Four wrongs squared*, minus two wrongs to the fourth power, divided by this formula, *do* make a right."

Problem: Are the complex numbers closed under exponentiation ? E.g., what is the value of i^i?